States of free product algebras and their integral representation

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1 Introduction

In his monograph [9], Hájek established theoretical basis for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named as Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [10], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert-Shields' Theorem [10] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and Product algebras respectively.

Within the setting of MFL, *states* were first introduced by Mundici [11] as maps averaging the truth-value in Łukasiewicz logic. In his work, states are functions mapping any MV-algebra $\mathbf{A}$ in the real unit interval $[0,1]$, satisfying a normalization condition and the additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, besides corresponding to convex combinations of valuations in Łukasiewicz propositional logic. However, states and probability measures were previously studied in [5] (see also [6, 13]) on Łukasiewicz tribes (σ-complete MV-algebras of fuzzy sets) as well as on other t-norm based tribes with continuous operations. MV-algebraic states have been deeply studied in recent years, as they enjoy several important properties and characterizations (see [8] for a survey).

One of the most important results of MV-algebraic state theory is Kroupa-Panti theorem [12, §10], a representation theorem showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one-to-one.

Many attempts of defining states in different structures have been made (see for instance [8, §8] for a short survey). In particular, in [2], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that results in corresponding to the integration of the truth value functions induced by Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Moreover, such states correspond to convex combinations of finitely many truth-value assignments.

The aim of this contribution is to introduce and study states for product logic, the remaining fundamental many-valued logic for which such a notion is still lacking. In particular, our axiomatization will result in characterizing Lebesgue integrals of the functions belonging to the free n-generated product algebra, i.e. the Lindenbaum algebra of product logic over n variables, with respect to Borel probability measures on $[0,1]^n$. In this sense, our states will correctly correspond to finitely additive probability measures in this context, and they will interestingly represent an axiomatization of the Lebesgue integral as an operator acting on product logic formulas. Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations.

2 States of free product algebras and their integral representation

Product algebras are BL-algebras satisfying two further equations:

$$x \wedge \neg x = 0 \text{ and } \neg\neg x \to ((y \cdot x \to z \cdot x) \to (y \to z)) = 1.$$

They constitute a variety that is the equivalent algebraic semantics for Product logic. In what follows, $\mathscr{F}_{\mathbb{P}}(n)$ will denote the free product algebra over n generators. We invite the interested reader to consult [1] and [7] for more details.

The functional representation theorem for free product algebras (cf. [1, Theorem 3.2.5]), shows that, up to isomorphism, every element of $\mathscr{F}_{\mathbb{P}}(n)$ is a *Product logic function*, i.e. $[0,1]$-valued function defined on $[0,1]^n$ associated to a product logic formula built over n propositional variables. These functions are for Product logic the equivalent counterpart of McNaughton functions for Łukasiewicz logic.

Next we introduce the notion of state of $\mathscr{F}_{\mathbb{P}}(n)$.

Definition 1. *A* state of $\mathscr{F}_{\mathbb{P}}(n)$ *is a map* $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$ *satisfying the following conditions:*

S1. $s(1) = 1$ *and* $s(0) = 0$*,*
S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$*,*
S3. If $f \leq g$*, then* $s(f) \leq s(g)$*,*

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

By the previous definition, it follows that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in [4]. However, if we compare Definition 1 with states of an MV-algebra, it is evident that, while for the case of MV-algebras the monoidal operation is directly involved in the axiomatization of states, the unique axiom that we impose and that, indirectly, involves the multiplicative connectives of product logic is S4.

Product logic functions in $\mathscr{F}_{\mathbb{P}}(n)$ are not continuous, unlike the case of free MV-algebras, and there are infinitely many, unlike the case for (finitely generated) free Gödel algebras. However, it is always possible to consider a *finite* partition of their domain, which depends on the Boolean skeleton of $\mathscr{F}_{\mathbb{P}}(n)$, upon which the restriction of each product function is continuous. By exploiting this fact, one can show the following integral representation theorem.

Theorem 1 (Integral representation). *For a $[0,1]$-valued map s on $\mathscr{F}_{\mathbb{P}}(n)$, the following are equivalent:*

(i) s is a state,
(ii) there is a unique Borel probability measure $\mu : \mathscr{B}([0,1]^n) \to [0,1]$ such that, for every $f \in \mathscr{F}_{\mathbb{P}}(n)$,

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

3 The state space and its extremal points

In the light of the previous Theorem 1, for n being a natural number, let us introduce the following notation: $\mathscr{S}(n)$ stands for the set of all states of $\mathscr{F}_{\mathbb{P}}(n)$, while $\mathscr{M}(n)$ denotes the set of all regular Borel probability measures on the Borel subsets of $[0,1]^n$. It is quite obvious that $\mathscr{S}(n)$ and $\mathscr{M}(n)$ are convex subsets of $[0,1]^{\mathscr{F}_{\mathbb{P}}(n)}$ and $[0,1]^{2^{[0,1]^n}}$ respectively, whence, by Krein-Milman Theorem they coincide with the convex hull of their extremal points. As for $\mathscr{M}(n)$ it is known that its extremal elements are Dirac measures, i.e., for each $x \in [0,1]^n$, those $\delta_x : 2^{[0,1]^n} \to [0,1]$ such that $\delta_x(B) = 1$ iff $x \in B$ and $\delta_x(C) = 0$ otherwise (see for instance [12, Cor. 10.6]).

Let $\delta : \mathscr{S}(n) \to \mathscr{M}(n)$ be the map that associates to every state its corresponding measure via Theorem 1. Thus, it is easy to prove that δ is bijective and affine. A direct consequence is that the extremal points of $\mathscr{S}(n)$, i.e., *extremal states* are mapped into extremal points of $\mathscr{M}(n)$, i.e. Dirac measures. Now, it is not hard to show that Dirac measures correspond univocally to the homomorphisms of $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, to the valuations of the logic, that hence are exactly the extremal states.

Theorem 2. *The following are equivalent for a state $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$*

1. *s is extremal;*
2. *$\delta(s)$ is a Dirac measure;*
3. *s is a product homomorphism.*

Thus, via Krein-Milman Theorem, we obtain the following:

Corollary 1. *For every $n \in \mathbb{N}$, the state space $\mathscr{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$.*

Acknowledgements: Flaminio and Godo acknowledges partial support by the FEDER/MINECO Spanish project RASO, TIN2015- 71799-C2-1-P.

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