On a three-valued logic to reason with prototypes and counterexamples and a similarity-based generalization

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Abstract. In this paper, the meaning of a vague concept α is assumed to be rendered through two (mutually exclusive) finite sets of prototypes and counterexamples. In the remaining set of situations the concept is assumed to be applied only partially. A logical model for this setting can be fit into the three-valued Łukasiewicz's logic $Ł_3$ set up by considering, besides the usual notion of logical consequence $\models$ (based on the truth preservation), the logical consequence $\models^{\leq}$ based on the preservation of all truth-degrees as well. Moreover, we go one step further by considering a relaxed notion of consequence to some degree $a \in [0,1]$, by allowing the prototypes (counterexamples) of the premise (conclusion) be a-similar to the prototypes (counterexamples) of the conclusion (premise). We present a semantical characterization as well as an axiomatization.

1 Introduction

A vague, in the sense of gradual, property is characterized by the existence of borderline cases; that is, objects or situations for which the property only partially applies. The aim of this paper is to investigate how a logic for vague concepts can be defined starting from the most basic description of a vague property or concept α in terms of a set of prototypical situations or examples $[\alpha^+] \subseteq \Omega$, where α definitely applies, and a set of counterexamples $[\alpha^-] \subseteq \Omega$, where α does not apply for sure. In this paper we will further assume to work with *complete* descriptions of this kind: that is, for each concept α, the remaining set of situations $\Omega \setminus ([\alpha^+] \cup [\alpha^-])$ will be those where we know α only partially applies to. Of course, to be in a consistent scenario, we will require there is no situation where α both fully applies and does not apply to, in other words, the constraint $[\alpha^+] \cap [\alpha^-] = \emptyset$ is always satisfied. In such a case, one lead to a three-valued framework, where for each situation $w \in \Omega$, the degree $app(w,\alpha)$ to which α applies at w (or, equivalently, the truth degree of the assertion "w is α") can be naturally defined as follows:

$$app(w,\alpha) = \begin{cases} 1, & \text{if } w \in [\alpha^+] \\ 0, & \text{if } w \in [\alpha^-] \\ 1/2, & \text{otherwise} \end{cases}$$

We want to emphasize that in this 3-valued model, the third value $1/2$ is not meant to represent ignorance about whether a concept applies or not to a situation, rather it is meant to represent that the concept only partially applies to a situation, or equivalently, that the situation is a borderline case for the concept (see [3] for a discussion on this topic).

The paper is structured as follows. After this short introduction, Section 2 is devoted to develop a logical approach to reason with vague concepts represented by examples and counterexamples based on the three-valued Łukasiewicz logic $Ł_3$. In Section 3 we show how by introducing a similarity relation into the picture one can define three kinds of graded notions of approximate logical consequence among vague propositions and we characterize them. Finally, in Section 4 we formally define a sort of graded modal logic to capture reasoning about the approximate consequences and prove completeness. We end up with some conclusions.

2 Three-valued logics to reason with examples and counterexamples

In our framework, we assume that we have evaluations e such that for atomic concepts α, $e(\alpha) = ([\alpha^+], [\alpha^-])$, providing a disjoint pair of examples and counterexamples. A first question is how this evaluation propagates to compound concepts. We consider a language with four connectives: conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) and implication ($\rightarrow$). Given $e(\alpha) = ([\alpha^+], [\alpha^-])$ and $e(\beta) = ([\beta^+], [\beta^-])$, the rules for $\wedge$, $\vee$ and $\neg$ seem clear as given follows:

$$e(\alpha \wedge \beta) = ([\alpha^+] \cap [\beta^+], [\alpha^-] \cup [\beta^-])$$
$$e(\alpha \vee \beta) = ([\alpha^+] \cup [\beta^+], [\alpha^-] \cap [\beta^-])$$
$$e(\neg\alpha) = ([\alpha^-], [\alpha^+])$$

The case for $\rightarrow$ is not that straightforward as above. Generalising the classical definition of material implication, one could take $\alpha \rightarrow \beta := \neg\alpha \vee \beta$, and hence

$$e(\neg\alpha \vee \beta) = ([\alpha^-] \cup [\beta^+], [\alpha^+] \cap [\beta^-]).$$

In that case, the framework turns out to be the one corresponding to the well-known Kleene's three-valued logic. However, it is also well-known that in Kleene's logic the interpretation of the intermediate value 1/2 is usually considered as ignorance. This makes it natural to claim that if it is not known whether w is an example or counterexample of both α and β, it remains unknown whether it is an example or counterexample of $\alpha \rightarrow \beta$. However, if $1/2$ is assumed to denote a borderline case, it is perfectly natural to consider, in that case, that w is an example of $\alpha \rightarrow \beta$. This small change in the framework amounts to move from Kleene's three-valued logic to Łukasiewicz's three-valued logic. In such a case, we have

$$e(\alpha \rightarrow \beta) = ([\alpha^-] \cup [\beta^+] \cup ([\alpha^\sim] \cap [\beta^\sim]), [\alpha^+] \cap [\beta^-]),$$

where we use the notation $[\gamma^\sim] = \Omega \setminus ([\gamma^+] \cup [\gamma^-])$.

Let us formalize this framework from a three-valued logic point of view. To do so, let Var denote a (finite) set of atomic concepts, or propositional variables, from which compound concepts (or formulas) are built using the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$. We will denote the set of formulas by $Fm_3(Var)$, in short Fm_3. Further, let Ω be the set of all possible situations, that we will identify with the set of all evaluations v of atomic concepts *Var* into the truth set $\{0,1/2,1\}$, that is $\Omega = \{0,1/2,1\}^{Var}$, with the following intended meaning: $v(\alpha) = 1$ means that v is an example of α (resp. v is a model of α in logical terms), $v(\alpha) = 0$ means that v is a counterexample of α (resp. v is a counter-model of α), and $v(\alpha) = 1/2$ means that v is borderline situation for α, i.e. it is neither an example nor a counterexample. According to the previous discussion, truth-evaluations v will be extended to compound concepts according to the semantics of 3-valued Łukasiewicz logic Ł$_3$, defined by following truth-tables:

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\rightarrow$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

$\neg$	
0	1
1/2	1/2
1	0

These truth-tables can also be given by means of the following truth-functions: for all $x,y \in \{0,1/2,1\}$, $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$, $x \rightarrow y = \min(1, 1-x+y)$ and $\neg x = 1-x$.

Notation For any concept φ we will denote by $[\varphi]$ the 3-valued (fuzzy) set of models of φ, i.e. $[\varphi] : \Omega \rightarrow \{0,1/2,1\}$ defined as $[\varphi](w) = w(\varphi)$. We will write $[\varphi] \leq [\psi]$ when $[\varphi](w) \leq [\psi](w)$ for all $w \in \Omega$.

In Ł$_3$, a strong conjunction and a strong disjunction connectives can be defined from $\rightarrow$ and $\neg$ as follows: $\varphi \otimes \psi := \neg(\varphi \rightarrow \neg\psi)$ and $\varphi \oplus \psi := \neg\varphi \rightarrow \psi$.[1] Actually, for each concept $\varphi \in Fm_3$, the connective $\otimes$ allows one to define three related *Boolean* concepts:

$$\varphi^+ := \varphi \otimes \varphi, \quad \varphi^- := (\neg\varphi) \otimes (\neg\varphi) = (\neg\varphi)^+, \quad \varphi^\sim := \neg\varphi^+ \wedge \neg\varphi^-,$$

with the following semantics:

$$\begin{array}{llll} w(\varphi^+) = 1 & \text{if} \;\; w(\varphi^+) = 1; & w(\varphi^+) = 0 & \text{otherwise;} \\ w(\varphi^\sim) = 1 & \text{if} \;\; w(\varphi^\sim) = 1/2; & w(\varphi^\sim) = 0 & \text{otherwise;} \\ w(\varphi^-) = 1 & \text{if} \;\; w(\varphi) = 0; & w(\varphi^-) = 0 & \text{otherwise;} \end{array}$$

and therefore $[\varphi^+], [\varphi^-], [\varphi^\sim]$ capture respectively the (classical) sets of examples, counterexamples and borderline cases of φ.

The usual notion of logical consequence in 3-valued Łukasiewicz logic is defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models \varphi \quad \text{if} \quad \text{for any evaluation } v,\ v(\varphi) = 1 \text{ for all } \varphi \in \Gamma, \text{ then } v(\psi) = 1.$$

It is well known that this consequence relation can be axiomatized by the following axioms and rule (see e.g. [2]):

(Ł1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
(Ł2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

[1] Actually, one could take $\rightarrow$ and $\neg$ as the only primitive connectives since $\wedge$ and $\vee$ can be defined from $\rightarrow$ and $\neg$ as well: $\varphi \wedge \psi = \varphi \otimes (\varphi \rightarrow \psi)$ and $\varphi \vee \psi = (\varphi \rightarrow \psi) \rightarrow \psi$.

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,
(Ł4) $(\varphi \vee \psi) \to (\psi \vee \varphi)$,
(Ł5) $\varphi \oplus \varphi \leftrightarrow \varphi \oplus \varphi \oplus \varphi$,
(MP) The rule of modus ponens: $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$.

This axiomatic system, denoted $Ł_3$, is strongly complete with respect to the above semantics; that is, for a set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$, where $\vdash$, the notion of proof for $Ł_3$, is defined from the above axioms and rule in the usual way.

Remark: In the sequel we will restrict ourselves on considerations about logical consequences from finite set of premises. In such a case, if $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ then it holds that $\Gamma \models \psi$ iff $\varphi_1 \wedge \ldots \wedge \varphi_n \models \psi$, and hence it will be enough to consider premises consisting of a single formula.

Lemma 1. *For all formulas φ, ψ, it holds that $\varphi \models \psi$ iff $[\varphi^+] \subseteq [\psi^+]$.*

This makes clear that $\models$ is indeed the consequence relation that preserves the examples of concepts. Similarly we can also consider the consequence relation that preserves counterexamples. Namely, one can contrapositively define a falsity-preserving consequence as:

$$\varphi \models^C \psi \text{ if } \neg\psi \models \neg\varphi\text{, that is, if for any evaluation } v\text{, } v(\psi) = 0 \text{ implies } v(\varphi) = 0.$$

Unlike classical logic, in 3-valued Łukasiewicz logic it is not the case that $\varphi \models \psi$ iff $\neg\psi \models \neg\varphi$. As we have seen that the former amounts to require $[\varphi^+] \subseteq [\psi^+]$, while the latter, as shown next, amounts to require $[\psi^-] \subseteq [\varphi^-]$. Clearly these conditions, in general, are not equivalent, except when φ and ψ do not have borderline cases, that is, when $[\varphi^+] \cup [\varphi^-] = [\psi^+] \cup [\psi^-] = \Omega$.

Lemma 2. *For all formulas φ, ψ, it holds that $\varphi \models^C \psi$ iff $[\psi^-] \subseteq [\varphi^-]$.*

Equivalently, $\varphi \models^C \psi$ holds iff for any evaluation $v \in \Omega$, $v(\varphi) \geq 1/2$ implies $v(\psi) \geq 1/2$, or in other words, $[\varphi^+] \cup [\varphi^\sim] \subseteq [\psi^+] \cup [\psi^\sim]$. Now we define the consequence relation that preserves both examples and counterexamples in the natural way.

Definition 1. $\varphi \models^{\leq} \psi$ *if* $\varphi \models \psi$ *and* $\varphi \models^C \psi$*, that is, if* $[\varphi^+] \subseteq [\psi^+]$ *and* $[\psi^-] \subseteq [\varphi^-]$.

Note that, for instance, $\varphi \models \varphi^+$ holds, while $\varphi \not\models^{\leq} \varphi^+$. Indeed, while the examples of φ and φ^+ are the same, the counterexamples of φ^+ include not only the counterexamples but also those borderline cases of φ.

From the above observations, we have these equivalent characterizations of $\models^{\leq}$.

Lemma 3. *For all formulas φ, ψ, the following conditions are equivalent:*

– $\varphi \models^{\leq} \psi$,
– $\models \varphi \to \psi$,
– $[\varphi] \leq [\psi]$,
– $[\varphi \to \psi] = \Omega$.

These characterizations justify the use of the superscript $\leq$ in the symbol of consequence relation. And indeed, the consequence relation $\models^{\leq}$ is known in the literature as the *degree-preserving* companion of $\models$, as opposed to the *truth-preserving* consequence $\models$, that preserves the truth-value '1' [1].

$\models^{\leq}$ can also be axiomatized by taking as axioms those of $Ł_3$ and the following two inference rules:

$$(Adj) : \frac{\varphi, \quad \psi}{\varphi \wedge \psi} \qquad (MPr) : \frac{\varphi, \quad \vdash \varphi \rightarrow \psi}{\psi}$$

The resulting logic is denoted by $Ł_3^{\leq}$, and its notion of proof is denoted by $\vdash^{\leq}$. Notice that (MPr) is a weakened version of modus ponens, called restricted modus ponens, since $\varphi \rightarrow \psi$ has to be a theorem of $Ł_3$ for the rule to be applicable.

As a summary of this section, we can claim that $Ł_3^{\leq}$ (or its semantical counterpart $\models^{\leq}$) provides a more suitable logical framework to reason about concepts described by examples and counterexamples than the usual three-valued Łukasiewicz logic $Ł_3$.

3 A similarity-based refined framework

In the previous section we have discussed a logic for reasoning about vague concepts described in fact as 3-valued fuzzy sets. A more fine grained representation, moving from 3-valued to $[0,1]$-valued fuzzy sets, can be introduced by assuming the availability of a (fuzzy) similarity relation $S : \Omega \times \Omega \rightarrow [0,1]$ among situations. Indeed, for instance, assume that all examples of φ are examples of ψ, but some counterexamples of ψ are not counterexamples of φ. Hence, we cannot derive that ψ follows from φ according to $\models^{\leq}$. However, if these counterexamples of ψ greatly resemble to counterexamples of φ, it seems reasonable to claim that ψ follows *approximately* from φ.

Actually, starting from Ruspini's seminal work [7], a similar approach has already been investigated in the literature in order to extend the notion of entailment in classical logic in different frameworks and using formalisms, see e.g. [6]. Here we will follow this line and propose a graded generalization of the $\models^{\leq}$ in the presence of similarity relation S on the set of 3-valued Łukasiewicz interpretations Ω, that allows to draw approximate conclusions.

Since, by definition $\varphi \models^{\leq} \psi$ if both $\varphi \models \psi$ and $\varphi \models^{C} \psi$, that is, if $[\varphi^{+}] \subseteq [\psi^{+}]$ and $[\psi^{-}] \subseteq [\varphi^{-}]$, it seems natural to define that ψ is an approximate consequence of φ to some degree $a \in [0,1]$ when every example of φ is similar (at least to the degree a) to some example of ψ, as well as every counterexample of ψ is similar (to at least to the degree a) to some counterexample of φ. In other words, this means that to relax $\models^{\leq}$ we propose to relax both $\models$ and $\models^{C}$. This idea is formalized next, where we assume that a $*$-similarity relation $S : \Omega \times \Omega \rightarrow [0,1]$ be given, satisfying the properties:

- $S(w,w') = 1$ iff $w = w'$,
- $S(w,w') = S(w',w)$,
- $S(w,w') * S(w',w'') \leq S(w,w'')$,

where $*$ is a t-norm operation. Moreover, for any subset $A \subset \Omega$ and value $a \in [0,1]$ we define its a-neighborhood as

$$A^{a} = \{w \in \Omega \mid \text{there exists } w' \in \Omega \text{ such that } S(w,w') \geq a\}.$$

Definition 2. *For any pair of formulas* φ, ψ *and for each degree* $a \in [0,1]$*, we define the consequence relations* $\models_a, \models_a^C$ *and* $\models_a^{\leq}$ *as follows:*

(i) $\varphi \models_a \psi$ *if for every* $w \in \Omega$ *such that* $w(\varphi) = 1$ *there exists* $w' \in \Omega$ *with* $S(w, w') \geq a$ *and* $w'(\psi) = 1$*. In other words,* $\varphi \models_a \psi$ *if* $[\varphi^+] \subseteq [\psi^+]^a$*.*
(ii) $\varphi \models_a^C \psi$ *if for every* $w \in \Omega$ *such that* $w(\psi) = 0$ *there exists* $w' \in \Omega$ *with* $S(w, w') \geq a$ *and* $w'(\varphi) = 0$*. In other words,* $\varphi \models_a^C \psi$ *if* $[\psi^-] \subseteq [\varphi^-]^a$*.*
(iii) $\varphi \models_a^{\leq} \psi$ *if both* $\varphi \models_a \psi$ *and* $\varphi \models_a^C \psi$ *i.e. if both* $[\varphi^+] \subseteq [\psi^+]^a$ *and* $[\psi^-] \subseteq [\varphi^-]^a$*.*

Taking into account that for any formula χ it holds $[(\neg\chi)^+] = [\chi^-]$, it is clear that $\models_a^C$ (and thus $\models_a^{\leq}$ as well) can be expressed in terms of $\models^a$.

Lemma 4. *For any formulas* φ *and* ψ*, the following conditions hold:*

- $\varphi \models_a^C \psi$ *iff* $\neg\psi \models_a \neg\varphi$
- $\varphi \models_a^{\leq} \psi$ *iff* $\varphi \models_a \psi$ *and* $\neg\psi \models_a \neg\varphi$*.*

The consequence relations $\models_a$ are very similar to the so-called approximate graded entailment relations defined in [4] and further studied in [6]. The main difference is that in [4] the authors consider classical propositions while in this paper we consider three-valued Łukasiewicz propositions. Nevertheless we can prove very similar characterizing properties for the $\models^a$'s. In the following theorem, for each evaluation $w \in \Omega$, $\overline{w}$ denotes the following proposition:

$$\overline{w} = (\bigwedge_{p \in X: w(p)=1} p^+) \wedge (\bigwedge_{p \in X: w(p)=1/2} p^\sim) \wedge (\bigwedge_{p \in X: w(p)=0} p^-).$$

So, $\overline{w}$ is a (Boolean) formula which encapsulates the complete description provided by w. Moreover, for every $w' \in \Omega$, $w'(\overline{w}) = 1$ if $w' = w$ and $w'(\overline{w}) = 0$ otherwise.

Theorem 1. *The following properties hold for the family* $\{\models_a : a \in [0,1]\}$ *of graded entailment relations on* $Fm_3 \times Fm_3$ *induced by a* $*$*-similarity relation* S *on* Ω*:*

(i) Nestedness: if $\varphi \models_a \psi$ *and* $b \leq a$*, then* $\varphi \models_b \psi$
(ii) $\models_1$ *coincides with* $\models$*, while* $\models \subsetneq \models_a$ *if* $a < 1$*. Moreover, if* $\psi \not\models \bot$*, then* $\varphi \models_0 \psi$ *for any* φ*.*
(iii) Positive-preservation: $\varphi \models_a \psi$ *iff* $\varphi^+ \models_a \psi^+$
(iv) $*$*-Transitivity: if* $\varphi \models_a \psi$ *and* $\psi \models_b \chi$ *then* $\varphi \models_{a*b} \chi$
(v) Left-OR: $\varphi \vee \psi \models_a \chi$ *iff* $\varphi \models_a \chi$ *and* $\psi \models_a \chi$
(vi) Restricted Right-OR: for all $w \in \Omega$*,* $\overline{w} \models_a \varphi \vee \psi$ *iff* $\overline{w} \models_a \varphi$ *or* $\overline{w} \models_a \psi$
(vii) Restricted symmetry: for all $w, w' \in \Omega$*,* $\overline{w} \models_a \overline{w'}$ *iff* $\overline{w'} \models_a \overline{w}$
viii) Consistency preservation: if $\varphi \not\models \bot$ *then* $\varphi \models_a \bot$ *only if* $a = 0$
(ix) Continuity from below: If $\varphi \models_a \psi$ *for all* $a < b$*, then* $\varphi \models_b \psi$

Conversely, for any family of graded entailment relations $\{\vdash_a : a \in [0,1]\}$ *on* $Fm_3 \times Fm_3$ *satisfying the above properties, there exists a *-similarity relation* S *such that* $\vdash_a = \models_a$ *for each* $a \in [0,1]$*.*

Proof. (Sketch) The proof follows the same steps than the one of [4, Th. 1] in the case of a classical propositional setting. The key points to take into account here are:

– it is easy to check that, for any formula $\varphi \in Fm_3$, φ^+ is logically equivalent in Ł$_3$ to the disjunction

$$\bigvee_{w\in\Omega: w(\varphi)=1} \overline{w}.$$

– $(\varphi \vee \psi)^+$ is logically equivalent to $\varphi^+ \vee \psi^+$.
– for every $w, w' \in \Omega$, $\overline{w} \models_a \overline{w'}$ iff $S(w, w') \geq a$.

For the converse direction, the latter property is used to define the corresponding similarity S for a family of consequence relations $\{\vdash_a : a \in [0,1]\}$ satisfying (i)-(ix) as $S(w, w') = \sup\{a \in [0,1] \mid \overline{w} \vdash_a \overline{w'}\}$. □

Taking into account Lemma 4, a sort of dual characterization for $\models_a^C$, that we omit, can easily be derived from the above one for $\models_a$. On the other hand, the above properties also indirectly characterize $\models_a^{\leq}$ in the sense that, in our finite setting, $\models_a$ (and thus $\models_a^C$ as well) can be derived from $\models_a^{\leq}$ as well as the following lemma shows.

Lemma 5. *For any $\varphi, \psi \in Fm_3$, we have that $\varphi \models_a \psi$ iff for every $w \in \Omega$ such that $w(\varphi) = 1$ there exists $w' \in \Omega$ such that $w(\psi) = 1$ and $\overline{w} \models_a^{\leq} \overline{w'}$.*

Proof. It directly follows from properties (iv) and (v) of Theorem 1, by checking that, for every $w \in \Omega$, $\overline{w} \models_a^{\leq} \overline{w'}$ iff $\overline{w} \models_a \overline{w'}$.

However, admittedly, the resulting characterization of $\models_a^{\leq}$ we would obtain using this lemma is not very elegant.

4 A logic to reason about graded consequences $\models_a$, $\models_a^C$ and $\models_a^{\leq}$

In this section we will define a Boolean (meta) logic LAC3 to reason about the graded entailments $\models_a$, $\models_a^C$ and $\models_a^{\leq}$. The idea is to consider expressions corresponding to $\varphi \models_a \psi$, $\varphi \models_a^C \psi$ and $\varphi \models_a^{\leq} \psi$ as the concerned objects of our logic, and then to use Theorem 1 to devise a complete axiomatics to capture the intended meaning of such expressions.

To avoid unnecessary complications, we will make the following assumption: all $*$-similarity relations S will take values in a finite set G of $[0,1]$, containing 0 and 1, and $*$ will be a given *finite* t-norm operation on G, that is, $(G, *)$ will be a finite totally ordered semi-group. In this way, we keep our language finitary and avoid the use of an infinitary inference rule to cope with Property (ix) of Theorem 1.

Our logic will be a two-tired logic, where at a first level we will have formulas and semantics of the 3-valued Łukasiewicz logic Ł$_3$ and at the second level we will have propositional classical logic CPC.

We start by defining the syntax of LAC3, with two languages:

– Language $\mathscr{L}_0$: built from a finite set of propositional variables $Var = \{p, q, r, \ldots\}$ and using Ł$_3$ connectives $\neg, \wedge, \vee, \rightarrow$. Other derived connectives are $\oplus$ and $\otimes$, defined as in Section 2. We will use $\top$ and $\bot$ as abbreviations for $p \rightarrow p$ and $\neg(p \rightarrow p)$ respectively, and φ^+ and φ^- as abbreviations of $\varphi \otimes \varphi$ and $(\neg\varphi)^+$ respectively.

- Language $\mathscr{L}_1$: atomic formulas of $\mathscr{L}_1$ are only of the form $\phi \succ_a^P \psi$, where ϕ, ψ are $\mathscr{L}_0$-formulas and $a \in G$, and compound $\mathscr{L}_1$-formulas are built from atomic ones with the usual Boolean connectives $\neg, \wedge, \vee, \rightarrow$.[2]
  Moreover, we will be using $\phi \succ_a^C \psi$ and $\phi \succ_a \psi$ as abbreviations of $\neg\psi \succ_a^P \neg\phi$ and $(\phi \succ_a^P \psi) \wedge (\phi \succ_a^C \psi)$ respectively.

The semantics is given by similarity Kripke models $M = (W, S, e)$ where W is a finite set of worlds, $S : W \times W \to G$ is a $*$-similarity relation, and $e : W \times Var \mapsto \{0, \frac{1}{2}, 1\}$ is a 3-valued evaluation of propositional variables in every world, which is extended to arbitrary $\mathscr{L}_0$-formulas using Ł$_3$ truth-functions. For every formula $\varphi \in \mathscr{L}_0$, we define: $[\varphi]_M : W \to \{0, 1/2, 1\}$ such that $w \mapsto e(w, \varphi)$, $[\varphi^+]_M = \{w \in W \mid e(w, \varphi) = 1\}$, and $[\varphi^-]_M = \{w \in W \mid e(w, \varphi) = 0\}$.

Each similarity Kripke model $M = (W, S, e)$ induces a function $e_M : \mathscr{L}_1 \to \{0, 1\}$, which is a (Boolean) truth evaluation for $\mathscr{L}_1$-formulas defined as follows:

- for atomic $\mathscr{L}_1$-formulas:
  $e_M(\phi \succ_a^P \psi) = 1$ if $[\phi^+]_M \subseteq ([\psi^+]_M)^a$, i.e., if $\min_{w \in [\phi^+]_M} \max_{w' \in [\psi^+]_M} S(w, w') \geq a$;
  $e_M(\phi \succ_a^P \psi) = 0$ otherwise.
- for compound formulas, use the usual Boolean truth functions.

Note that, by definition, $e_M(\phi \succ_a^C \psi) = 1$ iff $e_M(\neg\phi \succ_a^P \neg\psi) = 1$, and $e_M(\phi \succ_a \psi) = 1$ iff $e_M(\phi \succ_a^P \psi) = 1$ and $e_M(\phi \succ_a^C \psi) = 1$.

In the next lemma we list some useful properties of e_M.

Lemma 6. *The following conditions hold:*

- $e_M(\phi \succ_a^C \psi) = 1$ *iff* $[\psi^-]_M \subseteq ([\phi^-]_M)^a$
- $e_M(\phi \succ_a \psi) = 1$ *iff* $[\phi^+]_M \subseteq ([\psi^+]_M)^a$ *and* $[\psi^-]_M \subseteq ([\phi^-]_M)^a$
- $e_M(\phi \succ_1 \psi) = 1$ *iff* $[\phi]_M \leq [\psi]_M$
- $e_M((\phi \succ_1 \psi) \wedge (\psi \succ_1 \phi)) = 1$ *iff* $[\varphi]_M = [\psi]_M$, *iff* $[\phi \leftrightarrow \psi] = W$.

Now we define the notion of logical consequence in LAC3 for $\mathscr{L}_1$-formulas.

Definition 3. . *Let $T \cup \{\Phi\}$ be a set of $\mathscr{L}_1$-formulas. We say that Φ is a logically follows from T, written $T \models_{LAC3} \Phi$, if for every similarity Kripke model $M = (W, S, e)$, if $e_M(\Psi) = 1$ for every $\Psi \in T$, then $e_M(\Phi) = 1$ as well.*

Finally we propose the following axiomatization of LAC3.

Definition 4. *The following are the axioms for LAC3:*

(A1) *Axioms of CPC for $\mathscr{L}_1$-formulas*
(A2) $\phi \succ_1^P \psi$, *where ϕ, ψ are such that $\phi \models \psi$*
(A3) $\neg(\top \succ_1^P \rightarrow \bot)$
(A4) $(\phi \succ_a^P \psi) \rightarrow (\phi \succ_b^P \psi)$, *where $a \leq b$*

[2] Although we are using symbols $\wedge, \vee, \neg, \rightarrow$ for both formulas of $\mathscr{L}_0$ and $\mathscr{L}_1$, it will be clear from the context when they refer to Ł$_3$ or when they refer to Boolean connectives.

(A5) $(\phi \succ_1^P \psi) \to (\phi^+ \wedge \neg\psi^+ \succ_1^P \bot)$
(A6) $\neg(\psi \succ_1^P \bot) \to (\phi \succ_0^P \psi)$
(A7) $(\phi \succ_a^P \bot) \to (\phi \succ_1^P \bot)$
(A8) $\neg(\overline{w} \succ_1^P \bot) \wedge (\overline{w} \succ_a^P \overline{w'}) \ \to\ (\overline{w'} \succ_a^P \overline{w})$*, for* $w, w' \in \Omega$
(A9) $(\phi \succ_a^P \chi) \wedge (\psi \succ_a^P \chi) \ \to\ (\phi \vee \psi \succ_a^P \chi)$
(A10) $(\overline{w} \succ_a^P \phi \vee \psi) \ \to\ (\overline{w} \succ_a^P \phi) \vee (\overline{w} \succ_c^P \psi)$
(A11) $(\phi \succ_a^P \psi) \wedge (\psi \succ_b^P \chi) \to (\phi \succ_{a*b}^P \chi)$
(A12) $(\phi \succ_a^P \psi) \leftrightarrow \phi^+ \succ_a^P \psi^+$

The only rule of LAC3 is modus ponens. The notion of proof defined from the above axioms and rule will be denotes $\vdash_{LAC3}$*.*

Finally, we have the following soundness and completeness theorem for LAC3.

Theorem 2. *For any set* $T \cup \{\Phi\}$ *of* $\mathscr{L}_1$*-formulas, it holds that* $T \models_{LAC3} \Phi$ *if, and only if,* $T \vdash_{LAC3} \Phi$*.*

Proof. One direction is soundness, and it basically follows from Theorem 1. As for the converse direction, assume $T \not\vdash_{LAC3} \Phi$. The idea is to consider the graded expressions $\phi \succ_a^P \psi$ as propositional (Boolean) variables that are ruled by the axioms together with the laws of classical propositional logic CPC. Let Γ be the set of all possible instantiations of axioms (A1)-(A12). Then it implies that Φ does not follow from $T \cup \Gamma$ using CPC reasoning, i.e. $T \cup \Gamma \not\vdash_{CPC} \Phi$. By completeness of CPC, there exists a Boolean interpretation v such that $v(\Psi) = 1$ for all $\Psi \in T \cup \Gamma$ and $v(\Phi) = 0$. Now we will build a $*$-similarity Kripke model M such that $e_M(\Psi) = 1$ for all $\Psi \in T$ and $e_M(\Phi) = 0$. To do that we take Ω and define $S : \Omega \times \Omega \to G$ by

$$S'(w, w') = \max\{a \in G \mid v(\overline{w} \succ_a^P \overline{w'}) = 1\}.$$

By axioms (A2), (A8) and (A11), S is a $*$-similarity. Note that, by definition and Axiom (A4), $S(w, w') \geq a$ iff $v(\overline{w} \succ_a^P \overline{w'}) = 1$. Finally we consider the model $M = (\Omega, S, e)$, where for each $w \in \Omega$ and $p \in Var$, $e(w, p) = w(p)$. What remains is to check that $e_M(\Psi) = v(\Psi)$ for every LAC3-formula Ψ. It suffices to show that, for every $\phi, \psi \in \mathscr{L}_0$ and $a \in G$, we have $e_M(\phi \succ_a^P \psi) = v(\phi \succ_a^P \psi)$, that is, to prove that

$$v(\phi \succ_a^P \psi) = 1 \quad \text{iff} \quad \min_{w \in [\phi^+]_M} \max_{w' \in [\psi^+]_M} S(w, w') \geq a.$$

First of all, recall that for every ϕ, Ł$_3$ proves the equivalence $\phi^+ \leftrightarrow \vee_{w \in \Omega : w(\varphi) = 1} \overline{w}$, and by axioms (A12), (A9) and (A10), we have that LAC3 proves

$$\phi \succ_a^P \psi \ \leftrightarrow \ \bigwedge_{w \in \Omega : w(\phi) = 1} \ \bigvee_{w' \in \Omega : w'(\psi) = 1} \overline{w} \succ_a^P \overline{w'}.$$

Therefore, $v(\phi \succ_a^P \psi) = 1$ iff for all w in Ω such that $w(\phi) = 1$, there exists w' such that $w'(\psi) = 1$ and $v(\overline{w} \succ_a^P \overline{w'}) = 1$. But, as we have previously observed, $v(\overline{w} \succ_a^P \overline{w'}) = 1$ holds iff $S(w, w') \geq a$. In other words, we actually have $v(\phi \succ_a^P \psi) = 1$ iff $\min_{w \in [\phi^+]_M} \max_{w' \in [\psi^+]_M} S(w, w') \geq a$. This concludes the proof. □

5 Conclusions and future work

We have presented an approach towards considering graded entailments between vague concepts (or propositions) based on the similarity between both the prototypes and counterexamples of the antecedent and the consequent. This approach is a natural generalization of the Łukasiewicz's three-valued consequence ($\models^{\leq}$) that preserves truth-degrees. The provided axiomatization is for the operators $\succ_a^P$, which are based on prototypes only, while the operators $\succ_a$, based on both prototypes and counterexamples, can be naturally obtained as a derived operators in the system. To derive a complete axiomatic system directly for the operators $\succ_a$ is an issue under current investigation. Besides, we leave other interesting issues for further research. First, in this paper, we have assumed $app(\omega,\alpha)$ to be a three-valued concept, and to define $\models_a^{\leq}$ from $\models_a$ and $\models_a^C$ we have used a *conjunctive* aggregation of the two aspects of similarity, similarity among prototypes and similarity among counterexamples. Another approach could be to let $app(\omega,\alpha)$ to admit itself a finer distinction by defining $app^*(\omega,\alpha) = S(w,[\alpha]^+) \odot (1-S(w,[\alpha^-]))$ with $S(w,[\alpha^+]) = \max_{\omega' \in [\alpha^+]} S(\omega,\omega')$ and analogusly for $\mathrm{S}(w,[\alpha^-])$. Then the extent to which α entails β can be defined based on the relationship of $app^*(\omega,\alpha)$ and $app^*(\omega,\beta)$ considering all possible situations ω. This direction seems to have lots of challenges as $\odot$ might not be as simple as a conjunctive operation; also different notions of consequence can be worth exploring in the line of [5, 6, 8].

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