Similarity-based reasoning using prototypes and counterexamples

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Vague properties, in the sense of gradualness, are characterized by the existence of borderline cases; that is, objects or situations for which the property only partially applies. The aim of this paper is to investigate how a logic for vague concepts can be defined assuming that a vague concept α is given by a set of prototypical situations $[\alpha]^+ \subseteq \Omega$ where α definitely applies, as well as a set of counterexamples $[\alpha]^- \subseteq \Omega$ where α does not apply for sure. In this paper we will assume that this information is complete, that is, $[\alpha]^+$ is the whole set of prototypes and $[\alpha]^-$ is the whole set of counter-examples. This also means that the remaining set of situations $\Omega \setminus ([\alpha]^+ \cup [\alpha]^-)$ are those where α only partially applies. Of course, to be in a consistent scenario, we will require that $[\alpha]^+ \cap [\alpha]^- = \emptyset$. In such a case, one might think of a three-valued framework, where for each situation $w \in \Omega$ the degree to which α applies at w is defined as follows:

$$app(w,\alpha) = \begin{cases} 1, & \text{if } w \in [\alpha]^+ \\ 0, & \text{if } w \in [\alpha]^- \\ 1/2, & \text{otherwise} \end{cases}$$

This 3-valued vagueness model, where third value $1/2$ does not represent unknown but borderline (see [1] for a disucssion on this topic), is indeed very rough. A more refined model can be introduced by assuming the availability of a (fuzzy) similarity relation $S : \Omega \times \Omega \to [0,1]$ among situations. In such a case, for $w \in \Omega \setminus ([\alpha]^+ \cup [\alpha]^-)$ one can measure how close w is to some prototype of α, and on the one hand how close w is to some of its counterexamples.

$$S(w,[\alpha]^+) = \sup\{S(w,w') : w' \in [\alpha]^+\}$$
$$S(w,[\alpha]^-) = \sup\{S(w,w') : w' \in [\alpha]^-\}$$

Finally, to aggregate how much α applies to situation w, considering both the values, one can implement a commonsense rule like this one:

"The closer w is to some prototype and the farther is to any of the counterexamples, the **more α applies** to w"

Of course, one can think of different models of formalization following this rule; in principle one can think of a suitable aggregation operator $\otimes$ and define:

$$app^*(w,\alpha) = S(w,[\alpha]^+) \otimes (1 - S(w,[\alpha]^-))$$

This may in principle be appropriate as soon as $\otimes$ properly extends the above three-valued model in the sense that if $S(w,[\alpha]^+) = 1$ then $app^*(w,\alpha) = 1$ and if $S(w,[\alpha]^-) = 1$ then $app^*(w,\alpha) = 0$, and otherwise $0 < app^*(w,\alpha) < 1$. Assuming the similarity is strict, i.e. such that $S(w,w') = 1$ iff $w = w'$, a relevant example of such an aggregation operator, given in [4], is:

$$x \otimes y = \frac{y}{1-x+y},$$

but other operators may be suitable as well. Note that the mapping $\mu_\alpha : \Omega \to [0,1]$, defined as $\mu_\alpha(w) = app^*(w,\alpha)$, specifies a fuzzy set which can smoothly incorporate the finer distinctions of the apparent 3-valued nature of α. Or equivalently, one can also interpret $app^*(w,\alpha)$ as the degree to which α is satisfied by an interpretation, model or situation $w \in \Omega$.

Following the latter logical interpretation, the aim of this paper is to extend the approach used in [2, 3] (where only the values of $S(w,[\alpha]^+)$'s were considered) to define a logical framework to reason with fuzzy concepts given by a set of prototypes and counterexamples in the line of [4], but with some differences. The main difference is that we consider here as a working assumption that the base logic for our model of vague concepts based on prototypes and counter-examples is 3-valued Łukasiewicz logic $Ł_3$. Then, to refine such a logic, we will extend the language of $Ł_3$ with a modality $\Diamond$, and we will consider a Kripe-style semantics given by models $M = (W,e,S)$, where W is a set of worlds, $S : W \times W \to [0,1]$ is a similarity relation on worlds, and $e(w,\cdot) : V \to \{0,1/2,1\}$ is a $Ł_3$-valuation of variables. The evaluation will be extended to (non-nested) modal formulas by stipulating $e(w,\Diamond\alpha) = app^*(w,\alpha)$, where $[\alpha]^+ = \{w \in W \mid w(\alpha) = 1\}$ and $[\alpha]^- = \{w \in W \mid w(\alpha) = 0\}$. In this framework we plan to study different notions of graded entailment, as well as, to explore a Hilbert-style axiomatization and a proof system for the logical system.

References

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