Non-monotonic Fuzzy Measures and Intuitionistic Fuzzy Sets

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Abstract. Non-monotonic fuzzy measures induced by an intuitinistic fuzzy set are introduced. Then, using the Choquet integral with respect to the non-monotonic fuzzy measure, the weighted distance between two intuitionistic fuzzy sets is defined. As it will be shown here, under some conditions, the weighted distance coincides with the Hamming distance.

Keywords: Fuzzy measure, Non-monotonic fuzzy measure, Choquet integral, Intuitionistic fuzzy sets,Hamming distance.

1 Introduction

The so-called *intuitionistic fuzzy sets* were proposed by Atanassov [1, 2, 3] to have additional degrees of freedom when defining the membership values in a fuzzy set. Since then, the theory has been developed. Several new concepts and methods have been introduced and studied.

Fuzzy measures and fuzzy integrals are basic tools for decision modeling. Fuzzy integrals can be used to combine the information supplied by different information sources or to integrate the evaluation of different criteria. In this setting, fuzzy measures are used to represent the basic information about the sources (*e.g.*, their importance).

Although fuzzy measures are, typically, monotonic set functions on the unit interval, non-monotonic fuzzy measures have been also considered in the literature. See *e.g.* [8, 10, 11, 14]. In this paper we establish some relationships between non-monotonic fuzzy measures and intuitionistic fuzzy sets.

We show that non-monotonic fuzzy measures can be defined from intuitionistic fuzzy sets. Thus, given an intuitionistic fuzzy set, we will consider the fuzzy measure induced by it. Then, we will study some properties that establish relationships between intuitionistic fuzzy sets and (non-monotonic) fuzzy measures. The concept of bounded variation [4, 10, 11] (either positive or negative variation) plays a central role in such properties.

The structure of the paper is as follows. In Section 2, we present some preliminaries that are needed later on in this paper. In Section 3, we review the concepts

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of bounded variation and we present some results on the Choquet integrals of non-monotonic fuzzy measures. In Section 4, we introduce non-monotonic fuzzy measures induced by intuitionistic fuzzy sets. Using the Choquet integral with respect to the non-monotonic fuzzy measure, the weighted distance between two intuitionistic fuzzy sets is defined. We show that under some conditions, the weighted distance coincides with the Hamming distance. The paper finishes with some conclusions.

2 Preliminaries

In this section, we review some preliminary definitions and propositions on fuzzy measures and intuitionistic fuzzy sets. In the following, we will use the following notation. Let X be an universal set and let $\mathcal{X}$ be σ-algebra of X. That is, $(X, \mathcal{X})$ is a measurable space.

Definition 1. [12] Let $(X, \mathcal{X})$ be a measurable space. A *fuzzy measure* m is a real valued set function, $m : \mathcal{X} \longrightarrow R^+$ with the following properties;

(1) $m(\emptyset) = 0$
(2) $m(A) \leq m(B)$ whenever $A \subset B$, $A, B \in \mathcal{X}$.

We say that the triplet $(X, \mathcal{X}, m)$ is a fuzzy measure space if m is a fuzzy measure.

We will use $\mathcal{F}(X)$ to denote the class of non-negative measurable functions. That is,

$$\mathcal{F}(X) := \{f | f : X \to R^+, f : \text{measurable}\}$$

Definition 2. [5, 9] Let $(X, \mathcal{X}, m)$ be a fuzzy measure space. *The Choquet integral* of $f \in \mathcal{F}(X)$ with respect to m is defined by

$$(C) \int f dm = \int_0^\infty m_f(r) dr,$$

where $m_f(r) = m(\{x | f(x) \geq r\})$.

Definition 3. [6] Let $f, g \in \mathcal{F}(X)$. Then, we say that f and g are *comonotonic* if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x')$$

for $x, x' \in X$.

Proposition 4. *[6, 7] Let $(X, \mathcal{X}, m)$ be a fuzzy measure space. If $f, g \in \mathcal{F}(X)$ are comonotonic, then the Choquet integral is additive, that is,*

$$(C) \int (f + g) dm = (C) \int f dm + (C) \int g dm.$$

We say that the additivity of the Choquet integral, according to this property, is comonotonic additivity.

Next, we define an intuitionistic fuzzy set by Attanassov (for conciseness, we denote them by A-IFS).

Definition 5. *[1, 2, 3] An A-IFS (intuitionistic fuzzy set by Attanassov) A in X is defined by*

$$A := \{< x, \mu_A(x), \nu_A(x) > |x \in X\}$$

where $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *with*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

For each x*,* $\mu_A(x)$ *and* $\nu_A(x)$ *represent the degree of membership and degree of non-menbership of the element* $x \in X$ *to* $A \subset X$*, respectively. For each A-IFS, we define the intuitionistic fuzzy index by*

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x).$$

Suppose that X is a finite set, that is, $X := \{x_1, x_2, \ldots, x_n\}$. The Hamming distance between two A-IFS are proposed by Szmidt and Kacprzyk.

Definition 6. *[13] Let* $X := \{x_1, x_2, \ldots, x_n\}$ *be a finite universal set and* $A := \{< x, \mu_A(x), \nu_A(x) > |x \in X\}$*,* $B := \{< x, \mu_B(x), \nu_B(x) > |x \in X\}$ *be two A-IFS sets. Then,*

(1) *The Hamming distance* $d_{IFS}(A, B)$ *between* A *and* B *is defined by*

$$d_{IFS}(A, B) := \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$

(2) *The normalized Hamming distance* $l_{IFS}(A, B)$ *between* A *and* B *is defined by*

$$l_{IFS}(A, B) := \sum_{i=1}^{n} \frac{1}{2n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$

3 Non-monotonic Fuzzy Measure and Integral

Now, we turn into non-monotonic fuzzy measures and we show that the Choquet integral with respect to a non-monotonic fuzzy measure is comonotonically additive.

Definition 7. *[10, 11] Let* $(X, \mathcal{X})$ *be a measurable space. A non monotonic fuzzy measure is a real valued set function on* $\mathcal{X}$ *with* $m(\emptyset) = 0$*. We say that* $(X, \mathcal{X}, m)$ *is a non monotonic fuzzy measure space when* m *is a non monotonic fuzzy measure.*

Definition 8. *[4, 10] Let $(X, \mathcal{X}, m)$ be a non monotonic fuzzy measure space. Then, the positive variation $m^+(A)$ of m on the set $A \in \mathcal{X}$ is given by*

$$m^+(A) = \sup\{\sum_{i=1}^{n} \max\{m(A_i) - m(A_{i-1}), 0\}\}$$

where the sup is taken over all non decreasing sequences

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = A, A_i \in \mathcal{X}, i = 1, 2, \cdots n,$$

the negative variation $m^-(A)$ of m on the set $A \in \mathcal{X}$ is given by

$$m^-(A) = \sup\{\sum_{i=1}^{n} \max\{m(A_{i-1}) - m(A_i), 0\}\}$$

where the sup is taken over all non decreasing sequences

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = A, A_i \in \mathcal{X}, i = 1, 2, \cdots n$$

and the total variation $|m|(A)$ of m on the set $A \in \mathcal{X}$ is given by

$$|m|(A) = m^+(A) + m^-(A).$$

It is obvious from the definition above that

$$m(A) = m^+(A) - m^-(A)$$

for $A \in \mathcal{X}$.

We denote the variation $|m|(X)$ by $\|m\|$, and say that m is of bounded variation if $\|m\| < \infty$.

Definition 9. *The Choquet integral of a nonnegative measurable function $f \in \mathcal{F}(X)$ with respect to a non monotonic fuzzy measure m of bounded variation is defined by*

$$(C)\int f dm = \int_0^\infty m^+(\{x|f(x) \geq a\})da - \int_0^\infty m^-(\{x|f(x) \geq a\})da.$$

Since $m = m^+ - m^-$, the Choquet integral $C_m(f)$ is written by

$$C_m(f) := (C)\int f dm = \int_0^\infty m(\{x|f(x) \geq a\})da.$$

Let $f, g \in \mathcal{F}(X)$ be comonotonic. Then, the Choquet integrals with respect to m^+ and m^- are comonotonically additive. Therefore, the next proposition holds.

Proposition 10. *The Choquet integral with respect to a non-monotonic fuzzy measure m is comonotonically additive.*

Proof. Let $(X, \mathcal{X}, m)$ be a non monotonic fuzzy measure space and let $f, g \in \mathcal{F}(X)$ be comonotonic. Then,

$$\begin{aligned}(C)\int (f+g)dm &= (C)\int (f+g)dm^+ - (C)\int (f+g)dm^- \\ &= (C)\int fdm^+ + (C)\int gdm^+ - ((C)\int fdm^- + (C)\int gdm^-) \\ &= (C)\int fdm^+ - (C)\int fdm^- + (C)\int gdm^+ - (C)\int gdm^- \\ &= (C)\int fdm + (C)\int gdm. \square\end{aligned}$$

Let $\mathcal{A}$ be a chain of subsets of X, that is,

$$\mathcal{A} := \{A_i | i = 1, 2, \dots, n, A_i \subset X, \emptyset \subset A_1 \subset \cdots \subset A_n = X\}.$$

Since 1_{A_i} and 1_{A_j} are comonotonic for every $i, j = 1, 2, \dots n$ where 1_A is a characteristic function of A, we have

$$C_m(\sum_{i=1}^{n} a_i 1_{A_i}) = \sum_{i=1}^{n} a_i m(A_i)$$

for $a_i \geq 0$.

4 Non-monotonic Fuzzy Measure Induced by Intuitionistic Fuzzy Set

Let m be a non-monotonic fuzzy measure on X satisfying

$$0 \leq m(\{x\}) + m(X \setminus \{x\}) \leq 1, m(\{x\}) \geq 0, m(X \setminus \{x\}) \geq 0.$$

We can define an A-IFS $A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ by $\mu_A(x) := m(\{x\})$ and $\nu_A(x) := m(X \setminus \{x\})$. Conversely we can define a non-monotonic fuzzy measure from an A-IFS.

Definition 11. Let $A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ be an A-IFS. We define a non-monotonic fuzzy measure $\overline{m}_A : 2^X \to [0, 1]$ by

$$\overline{m}_A(B) = \begin{cases} 0 & \text{if } B = \emptyset \\ \sup_{y \in B} \mu_A(y) & \text{if } B \subsetneq X \setminus \{x\} \text{ for all } x \\ \nu(x) & \text{if for some } x, B = X \setminus \{x\} \\ \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) & \text{if } B = X, \end{cases}$$

and a non-monotonic fuzzy measure $\underline{m}_A : 2^X \to [0, 1]$ by

$$\underline{m}_A(B) = \begin{cases} 0 & \text{if } B = \emptyset \\ \sup_{y \in B} \nu_A(y) & \text{if } B \subsetneq X \setminus \{x\} \text{ for all } x \\ \mu(x) & \text{if for some } x, B = X \setminus \{x\} \\ \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \nu_A(y) & \text{if } B = X. \end{cases}$$

We say that $\overline{m_A}$ is a positive non-monotonic fuzzy measure induced by an intuitionistic fuzzy measure A and $\underline{m}_A$ is a negative non-monotonic fuzzy measure induced by an intuitionistic fuzzy measure A.

Let A and B be an A-IFS. Then, we define the following non-monotonic fuzzy measures for $C \subset X$:

$$(\overline{m}_A - \overline{m}_B)(C) := \overline{m}_A(C) - \overline{m}_B(C),$$

$$|\overline{m}_A - \overline{m}_B|(C) := |\overline{m}_A(C) - \overline{m}_B(C)|,$$

$$(\underline{m}_A - \underline{m}_B)(C) := \underline{m}_A(C) - \underline{m}_B(C),$$

$$|\underline{m}_A - \underline{m}_B|(C) := |\underline{m}_A(C) - \underline{m}_B(C)|.$$

The next lemma follows from the definition of a positive variation and a negative variation.

Lemma 12. *Let $A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ be an A-IFS and $\overline{m}_A$ be the positive non-monotonic fuzzy measure induced by the A-IFS A.*

$$\overline{m}_A^+(B) = \begin{cases} 0 & \text{if } B = \emptyset \\ \sup_{y \in B} \mu_A(y) & \text{if for some } x \in X, B \subsetneqq X \setminus \{x\} \\ \sup_{y \in C, C \subsetneqq B} \mu_A(y) & \text{if for some } x \in X, \quad B = X \setminus \{x\} \\ & \text{and } \sup_{y \in C, C \subsetneqq B} \mu_A(y) > \nu_A(x) \\ \nu_A(x) & \text{if for some } x \in X, \quad B = X \setminus \{x\} \\ & \text{and } \sup_{y \in C, C \subsetneqq B} \mu_A(y) \leq \nu_A(x) \\ \nu_A(x) & \text{if } B = X \text{ and for some } x \in X \\ & \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) \leq \nu_A(x) \\ \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) - \nu_A(x) & \\ \quad + \sup_{y \in C, C \subsetneqq X \setminus \{x\}} \mu_A(y) & \text{if } B = X \text{ and for some } x \in X, \\ & \sup_{y \in C, C \subsetneqq X \setminus \{x\}} \mu_A(y) \geq \nu(x) \\ \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) & \text{if } B = X \text{ and for some } x \in X, \\ & \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) \geq \nu(x) \\ & \text{and } \nu(x) \geq \sup_{y \in C, C \subsetneqq X \setminus \{x\}} \mu_A(y) \end{cases}$$

and

$$\overline{m}_A^-(B) = \begin{cases} 0 & \text{if } B \subsetneq X \setminus \{x\} \text{ for all } x \\ 0 & \text{if for some } x \in X, B = X \setminus \{x\} \\ & \text{and } \sup_{y \in C, C \subsetneq B} \mu_A(y) \leq \nu_A(x) \\ \sup_{y \in C, C \subsetneq B} \mu_A(y) - \nu_A(x) & \text{if for some } x \in X, B = X \setminus \{x\} \\ & \text{and } \sup_{y \in C, C \subsetneq B} \mu_A(y) \geq \nu_A(x) \\ \nu_A(x) - \inf x \in X \sup_{y \in X \setminus \{x\}} \mu_A(y) & \text{if } B = X, \text{ and for some } x \in X, \\ & \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) \leq \nu_A(x) \\ \sup_{y \in C, C \subsetneq X \setminus \{x\}} \mu_A(y) - \nu_A(x) & \text{if } B = X, \text{ and for some } x \in X, \\ & \sup_{y \in C, C \subsetneq X \setminus \{x\}} \mu_A(y) \geq \nu_A(x) \\ 0 & \text{if } B = X, \text{ and and and for some } x \in X, \\ & \inf_{x \in X} \sup_{y \in X \setminus \{x\}} \mu_A(y) \geq \nu(x) \\ & \text{and } \nu(x) \geq \sup_{y \in C, C \subsetneq X \setminus \{x\}} \mu_A(y) \end{cases}$$

Since $||\overline{m}_A|| = \overline{m}_A^+(X) + \overline{m}_A^-(X)$, we have the next proposition.

Proposition 13. *Let A be an A-IFS. Then, a positive (resp. negative) non-monotonic fuzzy measure induced by the A-IFS $\overline{m}_A$ (resp. $\underline{m}_A$) is of bounded variation.*

Since

$$\begin{aligned} |\overline{m}_A - \overline{m}_B| &= |\overline{m}_A^+ - \overline{m}_A^- + \overline{m}_B^+ - \overline{m}_B^-| \\ &\leq |\overline{m}_A^+| + |\overline{m}_A^-| + |\overline{m}_B^+| + |\overline{m}_B^-|, \end{aligned}$$

we have the next corollary.

Corollary 14. *Let A and B be intuitionistic fuzzy sets. Then, the fuzzy measures $\overline{m}_A - \overline{m}_B$, $|\overline{m}_A - \overline{m}_B|$, $\underline{m}_A - \underline{m}_B$ and $|\underline{m}_A - \underline{m}_B|$ are of bounded variation.*

It follows from Proposition 13 that we can define the Choquet integral with respect to a non-monotonic fuzzy measure induced by an A-IFS.

Definition 15. *Let A and B be an A-IFS, and $f,g,h \in \mathcal{F}(X)$. Then, the weighted distance ($wdist_{f,g,h}$) between A and B is defined by*

$$wdist_{f,g,h}(A,B) := C_{|\overline{m}_A - \overline{m}_B|}(f) + C_{|\underline{m}_A - \underline{m}_B|}(g) + C_{|(\overline{m}_A + \underline{m}_A) - (\overline{m}_B + \underline{m}_B)|}(h)$$

The weighted distance can be defined not only when X is a finite set, but also when X is infinite.

The next proposition immediately follows from this definition.

Proposition 16. *Let A, B and C be A-IFS, and* $f,g,h \in \mathcal{F}(X)$*.*

(1) $wdist_{f,g,h}(A,A) = 0$
(2) $wdist_{f,g,h}(A,B) = wdist_{f,g,h}(B,A)$
(3) $wdist_{f,g,h}(A,B) + wdist_{f,g,h}(B,C) \leq wdist_{f,g,h}(A,C)$
(4) *Suppose that* $f > 0, g > 0$ *and* $h > 0$*. Then,* $wdist_{f,g,h}(A,B) = 0$ *if and only if* $A = B$

In the following suppose that X is a finite set, that is, $X := \{x_1, x_2, \ldots, x_n\}$.

The next lemma follows from the definition of comonotonicity.

Lemma 17. *Let* $X := \{x_1, x_2, \ldots, x_n\}$*, and let* $f : X \to R$ *and* $g : X \to R$ *be comonotonic functions and* f *is one to one.*

(1) *If* $f(x_k) = \max_{x \in C} f(x)$*, then* $g(x_k) = \max_{x \in C} g(x)$ $(k = argmax_{x \in C} g(x))$ *for* $C \subset X$*.*
(2) *If* $f(x_k) = \min_{x \in X} \max_{y \in X \setminus \{x\}} f(y)$ *then* $g(x_k) = \min_{x \in X} \max_{y \in X \setminus \{x\}} g(y)$*.*

Proof. (1) Let $k = argmax_{x \in C} f(x)$. Since $|\{f(x)|x \in C\}| = n$, if $x \neq x_k$ then $f(x) < f(x_k)$.
Since f and g are comonotonic, $g(x) \leq g(x_k)$ for all $x \in C$.
Therefore $g(x_k) = max_{x \in C} g(x)$.

(2) Since $f(x_k) = \min_{x \in X} \max_{y \in X \setminus \{x\}} f(y)$, there exists $x_i \in X$ such that $f(x_k) = \max_{y \in X \setminus \{x_i\}} f(y)$. Then applying (1) we have $g(x_k) = \max_{y \in X \setminus \{x_i\}} g(y)$. Therefore $g(x_k) \geq \min_{x \in X} \max_{y \in X \setminus \{x\}} g(y)$.
Since for all $x \in X$

$$f(x_k) \leq \max_{y \in X \setminus \{x\}} f(y),$$

there exists $y \in X \setminus \{x_k\}$ such that $f(x_k) < f(y)$ since $y \neq x_k$. Then we have $g(x_k) \leq g(y)$, that is $g(x_k) \leq \max_{y \in X \setminus \{x_k\}} g(y)$. Therefore $g(x_k) \leq \max_{y \in X \setminus \{x\}} g(y)$ for all $x \in X$. that is, $g(x_k) \leq \min_{x \in X} \max_{y \in X \setminus \{x\}} g(y)$. □

Choosing the classes $\mathcal{C}, \mathcal{D}, \mathcal{E}$ of subsets of X suitably, using the previous lemma, we have the next proposition.

Proposition 18. *Let A and B be two A-IFS defined as follows:*

$$A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\},$$

$$B := \{< x, \mu_B(x), \nu_B(x) > | x \in X\},$$

μ_A *and* μ_B*,* ν_A *and* ν_B*,* $\mu_A + \nu_A$ *and* $\mu_A + \nu_B$ *are respectively comonotonic, and* $\mu_A, \mu_B, \mu_A + \nu_A$ *are one to one.*

Then, there exists a class $\mathcal{C} := \{C_i\}$ *,*$\mathcal{D} := \{D_i\}$*,* $\mathcal{E} := \{E_i\}$*, of subsets of X such that*

$$wdist_{f,g,h}(A,B) = \sum_{i=1}^{n} (a_i | \sup_{x \in C_i} \mu_A(x) - \sup_{x \in C_i} \mu_B(x)| + b_i | \sup_{x \in D_i} \nu_A(x) - \sup_{x \in D_i} \nu_B(x)| + c_i | \sup_{x \in E_i} \pi_A(x) - \sup_{x \in Ei} \pi_B(x)|)$$

where f, g, h *are linear combinations of characteristic functions and coefficients* $a_i \geq 0, b_i \geq 0, c_i \geq 0$*,that is,* $f := \sum_i a_i 1_{C_i}, g := \sum_i b_i 1_{D_i}, f := \sum_i c_i 1_{E_i}$.

Proof. Let $C'_i := \{x_1, x_2, \dots, x_i\}$, $f := \sum_{i=1}^n a'_i 1_{C'_i}$, $g := \sum_{i=1}^n b'_i 1_{C'_i}$ and $h := \sum_{i=1}^n c'_i 1_{C'_i}$ with $a'_i \geq 0, b'_i \geq 0, c_i \geq 0$.

Since each $1_{C'_i}$ and $1_{C'_j}$ are comonotonic,

$$\begin{aligned}
C_{|\overline{m}_A - \overline{m}_B|}(f) &= \sum_{i=1}^n C_{|\overline{m}_A - \overline{m}_B|}(a'_i 1_{C'_i}) \\
&= \sum_{i=1}^n a'_i |\overline{m}_A - \overline{m}_B|(C'_i) \\
&= \sum_{i=1}^n a'_i |\overline{m}_A(C'_i) - \overline{m}_B(C'_i)| \\
&= \sum_{i=1}^{n-2} a'_i |\sup_{x \in C'_i} \mu_A(x) - \sup_{x \in C'_i} \mu_B(x)| \\
&+ a'_{n-1} |\nu_A(x_n) - \nu_B(x_n)| + a'_n |\min_{x \in X} \max_{y \in X \setminus \{x\}} \mu_A(y) - \min_{x \in X} \max_{y \in X \setminus \{x\}} \mu_B(y)| \\
&= \sum_{i=1}^{n-2} a'_i |\sup_{x \in C'_i} \mu_A(x) - \sup_{x \in C'_i} \mu_B(x)| \\
&+ a'_{n-1} |\nu_A(x_n) - \nu_B(x_n)| + a'_n |\mu_A(x_{n-1}) - \mu_B(x_{n-1})|,
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
C_{|\underline{m}_A - \underline{m}_B|}(g) &= \sum_{i=1}^n C_{|\underline{m}_A - \underline{m}_B|}(b'_i 1_{D_i}) \\
&= \sum_{i=1}^n b'_i |\underline{m}_A - \underline{m}_B|(D_i) \\
&= \sum_{i=1}^n b'_i |\underline{m}_A(D_i) - \underline{m}_B(D_i)| \\
&= \sum_{i=1}^{n-1} b'_i |\sup_{x \in D_i} \nu_A(x) - \sup_{x \in D_i} \nu_B(x)| \\
&+ b'_{n-1} |\mu_A(x'_n) - \mu_B(x'_n)| + b'_n |\nu_A(x'_{n-1}) - \nu_B(x'_{n-1})|,
\end{aligned}$$

$$\begin{aligned}
C_{|(\overline{m}_A + \underline{m}_A) - (\overline{m}_B + \underline{m}_B)|}(f) &= \sum_{i=1}^n c'_i C_{|(\overline{m}_A + \underline{m}_A) - (\overline{m}_B + \underline{m}_B)|}(1_{E_i}) \\
&= \sum_{i=1}^n c'_i |(\overline{m}_A + \underline{m}_A) - (\overline{m}_B + \underline{m}_B)|(E_i) \\
&= \sum_{i=1}^n c'_i |(\overline{m}_A(E_i) + \underline{m}_A(E_i)) - (\overline{m}_B(E_i) + \underline{m}_B(E_i))|.
\end{aligned}$$

Changing coefficients a_i and the menber of the class $\mathcal{C}, \mathcal{D}, \mathcal{E}$ suitably, we have the concluding equality. □

Using the previous Proposition, we have the next proposition.

Proposition 19. *Let $A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ be an A-IFS, and let B be another A-IFS defined by $B := \{< x, \mu_B(x), \nu_B(x) > | x \in X\}$ such that μ_A, μ_B, ν_A, ν_B, $\mu_A + \nu_A$, $\mu_A + \nu_B$ are comonotonic, and $\mu_A, \nu_A, \mu_A + \nu_A$ are one to one.*

There exist functions f ,g,h on X such that

$$wdist_{f,g,h}(A, B) = \sum_{i=1}^{n} a_i(|\mu_A(x_i) - \mu_B(x_i)| + b_i|\nu_A(x_i)| - \nu_B(x_i)| + c_i|\pi_A(x) - \pi_B(x_i)|)$$

where $a_i \geq 0, b_i \geq 0, c_i \geq 0$.

Define f, g, h such that $a_i = b_i = c_i = 1$ for all i, we have the next corollary.

Corollary 20. *Let $A := \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ be an A-IFS, and let B another A-IFS defined by $B := \{< x, \mu_B(x), \nu_B(x) > | x \in X\}$ such that μ_A, μ_B, ν_A, ν_B, $\mu_A + \nu_A$, $\mu_A + \nu_B$ are comonotonic, and $\mu_A, \nu_A, \mu_A + \nu_A$ are one to one.*

Then, there exist functions f ,g,h on X such that

$$wdist_{f,g,h}(A, B) = d_{IFS}(A, B),$$

that is, wdist coincides with the Hamming distance.

Proof. Let $C_i := \{x_1, x_2, \ldots, x_i\}, i = 1, 2, \ldots, n$ and let $f := \sum_{i=1}^{n} 1_{C_i}$, $g := \sum_{i=1}^{n} 1_{C_i}$ and $h := \sum_{i=1}^{n} 1_{C_i}$.

Then, it follows from the proof of Proposition 12,

$$C_{|\overline{m}_A - \overline{m}_B|}(f) = \sum_{i=1}^{n-2} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_n) - \nu_B(x_n)| + |\mu_A(x_{n-1}) - \mu_B(x_{n-1})|,$$

$$C_{|\underline{m}_A - \underline{m}_B|}(f) = \sum_{i=1}^{n-2} |\nu_A(x_i) - \nu_B(x_i)| + |\nu_A(x_n) - \nu_B(x_n)| + |\mu_A(x_{n-1}) - \mu_B(x_{n-1})|,$$

and

$$C_{|\overline{m}_A + \underline{m}_A - \overline{m}_B - \underline{m}_B|}(f) = \sum_{i=1}^{n} |\pi_A(x_i) - \pi_B(x_i)|.$$ □

Define f, g, h such that the form of f is $f := \sum_{i=1}^{n}(1/n)1_{C_i}$, then we have the next corollary.

Corollary 21. *Let* $A := \{< x, \mu_A(x), \nu_A(x) > |x \in X\}$ *be an A-IFS, and let* B *be another A-IFS defined by* $B := \{< x, \mu_B(x), \nu_B(x) > |x \in X\}$*, and such that defined by* $B := \{< x, \mu_B(x), \nu_B(x) > |x \in X\}$ *such that* μ_A*,* μ_B*,* ν_A*,* ν_B*,* $\mu_A + \nu_A$ *,* $\mu_A + \nu_B$ *are comonotonic, and* $\mu_A, \nu_A, \mu_A + \nu_A$ *are one to one.*

Then, there exist functions f,g,h on X such that

$$wdist_{f,g,h}(A, B) = l_{IFS}(A, B).$$

5 Conclusions

In this paper we have proposed the definition of non-monotonic fuzzy measures in terms of intuitionistic fuzzy sets. We have seen that the Choquet integral of non-monotonic fuzzy measures permits to define the weighted distance between two intuitionistic fuzzy sets. We have also shown that under some conditions the weighted distance can be made equal to the Hamming distance.

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