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Axiomatizing Monoidal Logic: A Correction

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The propositional monoidal logic ML of U. HÖHLE, determined through an algebraic semantics given by the class of all residuated lattices, was (claimed to be) reaxiomatized in the recent monograph [5].

We show that this reaxiomatization is defective and prove the independence of one of U. HÖHLE's original axioms from the axioms in [5]. Additionally the axiomatization in [5] is completed.

Keywords: t-norm, residuated lattices, monoidal logic

1. INTRODUCTION

Fruitful sources for inspiration in many-valued logic have in recent years been the theories of fuzzy sets and fuzzy logic, together with their applications. In pure logic, an important stream of development caused by this inspiration has been the extension of the class of infinite valued systems with the real unit interval [0,1] as truth degree set, and has also been their unification via the consideration of t-norms as truth degree functions for (basic) conjunction connectives.

These t-norms are isotone binary operations in the unit interval which are commutative, associative, and have 1 as a neutral element, i.e. they make the unit interval into a commutative ordered monoid. Standard examples of t-norms are the truth degree functions of the (suitable) conjunction connectives in the infinite valued logics of Łukasiewicz and of Gödel, and in the product logic [7].

In the case that such a t-norm $*$ is a left continuous function (in both of its arguments) it is uniquely tied with a left adjoint operation $\rightarrowtail$, also called the *residuation* of $*$, such that the adjointness condition

$$a * b \leq c \qquad \Leftrightarrow \qquad a \leq b \rightarrowtail c$$

holds true for all $a, b, c \in [0,1]$, or equivalently, that one always has

$$b \rightarrowtail c = \max\{x \mid x * b \leq c\}.$$

With this residuation as truth degree function of an implication connective, with the function $-x =_{def} x \rightarrowtail 0$ as truth degree function of a negation connective, and with the degree 1 as the only designated truth degree, and finally also with further "weak" conjunction and disjunction connectives with min and max as their truth degree functions, each left continuous t-norm determines a particular infinite valued propositional logic.

Standard examples again are the infinite valued logics of Łukasiewicz and of Gödel, as well as the product logic.

The axiomatization problem for such t-norm-based residuated logics was in the last years successfully treated mainly not for particular t-norms, but for whole classes τ of t-norms: which means that adequate axiomatizations have been given for classes of all propositional formulas logically valid w.r.t. such a class τ, i.e. logically valid in all t-norm-based logics with t-norms from such classes τ.

These approaches were made via some algebraic semantics, i.e. logical validity was not only determined by such a class τ of t-norms, it was determined by a suitable superclass of algebraic structures. In the most essential cases these have been classes of residuated lattices. These approaches hence generalize the linear ordering in the unit interval to lattice orderings.

The first approach toward such a t-norm related residuated logic, which arose out of the consideration of t-norms as conjunction connectives, was the *monoidal logic* ML of U. HÖHLE [8], characterized through the class of all residuated lattices. From a proof theoretical point of view, however, this logic had already been discussed in [10] (denoted H_{BCK}) and also in [1] (where was considered a contraction-less intuitionistic logic with conjunction and fusion).

The Hilbert type axiomatization of U. HÖHLE [8] has the following axiom schemata

(Ax$^*_{ML}$1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ax$^*_{ML}$2) $\varphi \,\&\, \psi \to \varphi$,

(Ax$^*_{ML}$3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$,

(Ax$^*_{ML}$4) $(\varphi \to (\psi \to \chi))\to(\varphi \,\&\, \psi \to \chi)$,

(Ax$^*_{ML}$5) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,

(Ax$^*_{ML}$6) $\varphi \wedge \psi \to \varphi$,

(Ax$^*_{ML}$7) $\varphi \to \varphi \vee \psi$,

(Ax$^*_{ML}$8) $\psi \to \varphi \vee \psi$,

(Ax$^*_{ML}$9) $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$,

(Ax$^*_{ML}$10) $\varphi \wedge \psi \to \psi$,

(Ax$^*_{ML}$11) $(\varphi \,\&\, \psi) \,\&\, \chi \to \varphi \,\&\, (\psi \,\&\, \chi)$,

(Ax$^*_{ML}$12) $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to \psi \wedge \chi))$,

(Ax$^*_{ML}$13) $\varphi \,\&\, \neg\, \varphi \to \psi$,

(Ax$^*_{ML}$14) $(\varphi \to \varphi \,\&\, \neg\, \varphi) \to \neg\, \varphi$,

and has as its (only) inference rule the rule of detachment (w.r.t. the implication connective $\to$).

Motivated by t-norm related considerations on fuzzy sets and fuzzy logic, later on P. HÁJEK [6] considered the *basic t-norm logic* BL, characterized through the class of all prelinear and divisible residuated lattices.[1] And even later F. ESTEVA / L. GODO [3] considered the *monoidal t-norm logic* MTL, characterized through the class of all prelinear residuated lattices, i.e. through the class of all residuated lattices which additionally satisfy the condition

$$(a \to b) \vee (b \to a) = 1.$$

[1] All these particular mathematical notions are e.g. explained in [5].

Meanwhile it has been shown in [2] that BL is really the logic of all continuous t-norms, and in [9] that MTL is really the logic of all left continuous t-norms.

2. AN INDEPENDENCE RESULT

There was a certain difference in the original axiomatizations of monoidal logic ML, and of monoidal t-norm logic MTL as well as of basic t-norm logic BL. This difference was mainly coming from the fact that the basic (propositional) languages were chosen differently: the primitive connectives for ML in [8] had been $\langle \wedge, \vee, \& \rightarrow, \neg \rangle$, but the primitive connectives for MTL and BL were $\langle \wedge, \&, \rightarrow, 0 \rangle$, and $\langle \&, \rightarrow, 0 \rangle$, respectively.

To unify matters, the first of us had given in [4], and again in [5], a different axiomatization for ML in the basic vocabulary $\langle \wedge, \vee, \&, \rightarrow, 0 \rangle$, consisting of the rule modus ponens and the axiom schemata

($\mathrm{Ax_{ML}}1$) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

($\mathrm{Ax_{ML}}2$) $\varphi \,\&\, \psi \rightarrow \varphi$,

($\mathrm{Ax_{ML}}3$) $\varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi$,

($\mathrm{Ax_{ML}}4$) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \psi \rightarrow \chi)$,

($\mathrm{Ax_{ML}}5$) $(\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,

($\mathrm{Ax_{ML}}6$) $\varphi \wedge \psi \rightarrow \varphi$,

($\mathrm{Ax_{ML}}7$) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$,

($\mathrm{Ax_{ML}}8$) $\varphi \,\&\, (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$,

($\mathrm{Ax_{ML}}9$) $0 \rightarrow \varphi$,

($\mathrm{Ax_{ML}}10$) $\varphi \rightarrow \varphi \vee \psi$,

($\mathrm{Ax_{ML}}11$) $\psi \rightarrow \varphi \vee \psi$,

($\mathrm{Ax_{ML}}12$) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$.

Unfortunately, in the proof that the new axioms for ML suffice to derive the ones given in [8] there remained an error unnoticed.

In [5], the proof of ($\mathrm{Ax^{*}_{ML}}12$), i.e. of the formula

$$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi)) \,, \tag{1}$$

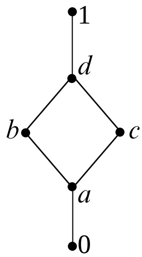
suffers from an unmotivated exchange of indices. But it is not only the actual proof which has a defect: The claim to have with axiom system Ax_{ML} an equivalent axiomatization for Ax^*_{ML} has to be updated because formula (1) is actually not derivable from the axiom system Ax_{ML}, as the following proposition shows.

Proposition 1 *The formula*

$$(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to \psi \wedge \chi))$$

is independent of the axiom system ($Ax_{ML}1$) *to* ($Ax_{ML}12$)*, using modus ponens as the only inference rule.*

Proof: Suppose A is the set $\{0, a, b, c, d, 1\}$ and define in it the lattice and the operations given below.[2]



&	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	d	d	1	d	1	1
c	d	d	d	1	1	1
d	d	d	d	d	1	1
1	0	a	b	c	d	1

It is easy to check that $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, 0 \rangle$ is a residuated lattice. Define the operation Δ as

$$x \,\Delta\, y := \begin{cases} x \wedge y, & \text{if } \{x, y\} \neq \{b, c\} \\ 0 & , \text{ if } \{x, y\} = \{b, c\} \end{cases}$$

Now consider the structure $\boldsymbol{B} = \langle A, \Delta, \vee, \&, \to, 0 \rangle$ together with the lattice ordering $\leqq$ from $\boldsymbol{A}$, and consider $\boldsymbol{B}$-evaluations of our language. It is straightforward to verify that $\boldsymbol{B}$ becomes a model of the axiom system ($Ax_{ML}1$) to ($Ax_{ML}12$): only the axioms ($Ax_{ML}6$), ($Ax_{ML}7$) and ($Ax_{ML}8$) are affected by this reinterpretation of the weak conjunction $\wedge$, however one has $y \,\Delta\, x = x \,\Delta\, y \leqq x$ and $x \,\&\, (x \to y) \leqq x \,\Delta\, y$ over the elements of A which

[2] For simplicity we use (mainly) the same symbols for the logical connectives and their counterparts in the following algebraic structures.

means that also ($\mathrm{Ax_{ML}}6$), ($\mathrm{Ax_{ML}}7$) and ($\mathrm{Ax_{ML}}8$) remain ***B***-valid. Furthermore modus ponens remains a ***B***-valid inference rule. This means that every formula H derivable from the axiom system $\mathrm{Ax_{ML}}$ using modus ponens is ***B***-valid, i.e. satisfies $e(H) = 1$ for all evaluations e in ***B***. However,

$$(a \to b) \to ((a \to c) \to (a \to (b \,\Delta\, c))) = d.$$

This means that (1) is not derivable from the axiom system ($\mathrm{Ax_{ML}}1$) to ($\mathrm{Ax_{ML}}12$) using modus ponens.[3] □

3. AXIOMATIZING MONOIDAL LOGIC

As a solution of the axiomatization problem under consideration here, formula (1) has to be added to the system ($\mathrm{Ax_{ML}}1$) to ($\mathrm{Ax_{ML}}12$) of axiom schemata. But then, axiom ($\mathrm{Ax_{ML}}8$) becomes derivable.

Proposition 2 *An adequate axiomatization of ML is provided by the axiom schemata* ($\mathrm{Ax_{ML}}1$) *to* ($\mathrm{Ax_{ML}}7$) *and* ($\mathrm{Ax_{ML}}9$) *to* ($\mathrm{Ax_{ML}}12$) *together with* ($\mathrm{Ax^{*}_{ML}}12$).

Proof: One derives $\varphi \to (\psi \to \varphi)$ immeditely from ($\mathrm{Ax_{ML}}2, 5$). And one derives from ($\mathrm{Ax_{ML}}4, 5$) via ($\mathrm{Ax_{ML}}1, 3$) the formula $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$. Both formulas together allow to derive $\varphi \to \varphi$.

Therefore from (1), i.e. from ($\mathrm{Ax^{*}_{ML}}12$) one derives $(\varphi \to \chi) \to (\varphi \to \varphi \wedge \chi)$, which finally gives ($\mathrm{Ax_{ML}}8$) via ($\mathrm{Ax_{ML}}3, 4$). □

The further axiomatization results mentioned in [5] are not influenced by this addition because formula (1) becomes provable as soon as the prelinearity axiom

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \qquad (2)$$

is present. A proof of (1) using (2) is easy to obtain from the proof of lemma 2.2.9 (12) in [6]. And (2) is available as soon as one is considering the monoidal t-norm logic MTL.

[3] This also means that ***B*** is not a residuated lattice.

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