On the hyperreal state space

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Abstract

The aim of this work is to study a class of non-archimedean valued measures on MV-algebras. We call them *hyperreal states* and their definition naturally arise from (the uniform version of) Di Nola representation theorem for MV-algebras (cf [5, 6]): for any MV-algebra $A = (A, \oplus, \neg, \top, \bot)$ there exists a ultrafilter $\mathfrak{U}$ on the cardinality of A such that A embeds into $({}^*[0,1]_{\mathfrak{U}})^{Spec(A)}$ (where as usual $Spec(A)$ denotes the space of prime ideals of A). Therefore, if A is any MV-algebra, there exists a non-archimedean extension ${}^*[0,1]_{\mathfrak{U}}$ of the real unit interval $[0,1]$ such that every element a of A can be regarded as a function $f_a : Spec(A) \to {}^*[0,1]_{\mathfrak{U}}$. Since MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic, Di Nola's theorem states that formulas of Łukasiewicz calculus are visualized as black and white pictures printed by a palette of *infinitesimals* grey levels ([9, §2]).

As it is well known [3], the proper subclass of *semisimple* MV-algebras can be characterized, by Chang and Belluce theorem (cf. [2, 1]), as algebras of *real-valued* functions: up to isomorphisms every semisimple MV-algebra A is an algebra of $[0,1]$-valued functions defined over the space of maximal ideals $\mathcal{M}(A)$ of A. Therefore, following the above metaphor, if we interpret formulas of Łukasiewicz calculus into a semisimple MV-algebra, then by Chang and Belluce theorem there are no infinitesimals grey levels, and hence the palette used to print (i.e. interpret) the propositions, only has *real* grey levels (cf. [9, §2]).

States on MV-algebras have been introduced by Mundici in [9]: for every MV-algebra A, *state* on A is a map $s : A \to [0,1]$ that is, normalized and additive[1]. Therefore states are real valued maps defined on (possibly) hyperreal-valued functions, and hence they do no preserve, the non-archimedean structure of the MV-algebra they are defined over.

[1]We refer the reader to [9, Definition 2.1].

Therefore we introduce *hyperreal states* to support the intuition that, if an MV-algebra A do have infinitesimal elements (and hence, is not semisimple), then a *state* on A should preserve the same grey-level palette also to *measure* its elements. Those are defined as follows: let A be an MV-algebra, then a *hyperreal state* on A is a map $s : A \to {}^*[0,1]_{\mathfrak{U}}$ satisfying:

(i) $s(\top) = 1$ (normalization),

(ii) whenever $x \odot y = \bot$, $s(x \oplus y) = s(x) + s(y)$ (additivity).

We denote by $\mathcal{HS}(A)$ the set of all hyperreal states on A. Consider the following: for every A, let $\mathfrak{p}$ be a prime filter in $Spec(A)$. The quotient $A/\mathfrak{p}$ is linearly ordered (call f the canonical homomorphism of A into $A/\mathfrak{p}$), and $A/\mathfrak{p}$ embeds into ${}^*[0,1]_{\mathfrak{U}}$ via a map g (where $\mathfrak{U}$ is referred to A). Then the map $s_{\mathfrak{p}} = g \circ f : A \to {}^*[0,1]_{\mathfrak{U}}$ is a hyperreal state of A, whence, $\mathcal{HS}(A)$ is non empty.

Let ${}^*[0,1]$ be any non-archimedean extension of the real unit interval $[0,1]$. Let us denote by sh is the *shadow map* from ${}^*[0,1]$ to $[0,1]$ mapping every $x \in {}^*[0,1]$ into that unique real $sh(x)$ such that the distance $|x - sh(x)|$ is infinitesimal. Then hyperreal states provide a generalization of states [9] in the following sense:

Theorem 1. *if A is a semisimple MV-algebra (and hence A is an MV-algebra of $[0,1]$-valued functions), every hyperreal state on A actually is a state. Moreover if A is not semisimple and s is a hyperreal state on A, then the composition $sh \circ s$ is a state on A.*

In order to study the geometric properties of the hyperreal state space $\mathcal{HS}(A)$ as a subspace of ${}^*[0,1]_{\mathfrak{U}}^A$, we noticed that there is no a standard way to extend the usual interval topology of reals to every unit interval ${}^*[0,1]$ and keeping the space to be locally convex. The S-topology (cf. [8]) suffices our purposes of making ${}^*[0,1]$ a locally convex space, but unfortunately it does not preserve the property of being Hausdorff. The hyperreal state space $\mathcal{HS}(A)$ can hence be regarded as a compact subspace of the locally convex space ${}^*[0,1]_{\mathfrak{U}}^A$. Then Krein-Milman theorem can be applied to show that $\mathcal{HS}(A)$ coincides with the closure of its extremal points $\mathrm{ext}(\mathcal{HS}(A))$. As it is well known (see [9, Theorem 2.5]) the extremal states of A are homomorphisms of A into $[0,1]$. As regards to hyperreal states we proved the following:

Theorem 2. *Let A be an MV-algebra, and let $s \in \mathcal{HS}(A)$. Then s is an extremal hyperreal state iff $sh \circ s$ is extremal state.*

The above theorem, together with [9, Theorem 2.5] states that a hyperreal state s on an MV-algebra A is extremal iff $sh \circ s$ is a homomorphism of A into $[0,1]$. Therefore the extremal hyperreal states on A and the space $Spec(A)$ of prime filters of A, are related as follows: for every prime filter $\mathfrak{p} \in Spec(A)$, the map

$$a \in A \overset{f}{\mapsto} a/\mathfrak{p} \in A/\mathfrak{p} \overset{g}{\mapsto} {}^*\alpha \in {}^*[0,1]_{\mathfrak{U}}, \tag{1}$$

where f and g are as above, is a homomorphism, and hence is an extremal hyperreal state of A because $sh \circ f \circ g$ is a homomorphism of A into $[0,1]$. Conversely, for every homomorphism $s : A \to {}^*[0,1]_{\mathfrak{U}}$ (and hence

$sh \circ s \in \mathrm{ext}\mathcal{S}(A))$, $\mathfrak{p}_s = \{x \in A : h(x) = 1\}$ is a prime filter of A. In fact, since s is a homomorphism, its kernel is an ideal of A, whence $\mathfrak{p}_s$ is a filter. Moreover, if $x \wedge y \in \mathfrak{p}_s$, then $s(x \wedge y) = 0$, and hence $s(x) \wedge s(y) = 0$. Since ${}^*[0,1]_{\mathfrak{U}}$ is totally ordered, $s(x) = 0$, or $s(y) = 0$, i.e. $x \in \mathfrak{p}_s$, or $y \in \mathfrak{p}_s$, whence $\mathfrak{p}_s$ is prime filter.

In the case of states, the space of extremal states $\mathrm{ext}(\mathcal{S}(A))$, and the space $\mathcal{M}(A)$ of maximal filters of A are homeomorphic. This parallelism seems not to be recoverable when we move to hyperreal states, and hence when we consider the space $Spec(A)$ (endowed with spectral topology) and $\mathrm{ext}(\mathcal{HS}(A))$. Actually, $Spec(A)$ is a T_0 space, while on the other hand ${}^*[0,1]$ endowed with the S-topology is not T_0 (cf. [4, Theorem 1.4]).

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