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On some properties of quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras. Part II

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Abstract The present paper is a sequel to Paoli F, Ledda A, Giuntini R, Freytes H (On some properties of QMV algebras and $\sqrt{'}$QMV algebras, submitted). We provide two representation results for quasi-MV algebras in terms of MV algebras enriched with additional structure; we investigate the lattices of subvarieties and subquasivarieties of quasi-MV algebras; we show that quasi-MV algebras, as well as cartesian and flat $\sqrt{'}$ quasi-MV algebras, have the amalgamation property.

1 Introduction

Quasi-MV algebras (for short, QMV algebras) were introduced in Ledda et al. (2006) in connection with quantum computation— namely, in an attempt to provide a convenient abstraction of the algebra over the set of all density operators of the Hilbert space $\mathbb{C}^2$, endowed with a suitable stock of quantum logical gates. Independently of their original quantum computational motivation, QMV algebras present an additional, purely algebraic, motive of interest as generalisations of MV algebras to the semisubtractive (in the sense of Salibra 2003) but not point regular case. Later, $\sqrt{'}$ *quasi-MV algebras* (for short, $\sqrt{'}$ QMV algebras) were introduced as term expansions of QMV algebras by an operation of square root of the inverse (Giuntini et al. 2007). The above referenced papers contain the basics of the structure theory for these varieties, including appropriate standard completeness theorems w.r.t. the algebras over the complex numbers which constituted the motivational starting point of the whole investigation. In a subsequent paper (Paoli et al. submitted) the algebraic properties of QMV algebras and $\sqrt{'}$ QMV algebras were investigated in greater detail: the finite model property and the congruence extension property were established for both varieties; characterisations of the QMV term reducts and subreducts of $\sqrt{'}$ QMV algebras were offered; suitable descriptions of semisimple $\sqrt{'}$ QMV algebras and of free algebras in both varieties were given.

In the present paper, we try to pile up a few more results of the same kind. In the next section, we give two representations of QMV algebras in terms of labelled MV algebras. In Sect. 3, we investigate the lattice of subvarieties of QMV algebras. In Sect. 4 we compare the notion of QMV ideal introduced in Ledda et al. (2006) with Ursini's universal algebraic concept of ideal, while in Sect. 5 we single out appropriate sets of generators for $\mathbb{QMV}$ and $\sqrt{'}\mathbb{QMV}$ as quasivarieties. Finally, in Sect. 6 we establish the amalgamation property for QMV algebras on the one hand, and flat and cartesian $\sqrt{'}$ QMV algebras on the other.

With an eye to shrinking the paper down to an acceptable length, we assume familiarity with both the content and the notation of Ledda et al. (2006); Giuntini et al. (2007); Paoli et al. (submitted). As further notational conventions, we hereafter denote:

- by $\mathbf{F}_{nm}$ $(n, m \geq 0)$ the flat QMV algebra with $n + 1$ fixpoints of the inverse (including 0) and m elements different from their own inverses;

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- by $\mathbf{F}_{nmp}$ $(n, m, p \geq 0)$ the flat $\sqrt{'}$ QMV algebra with $n+1$ fixpoints under $\sqrt{'}$ (including 0), m fixpoints under $'$ which are not fixpoints under $\sqrt{'}$, and p elements which are not fixpoints under either operation.

By means of this notation we are in a position to unambiguosly refer to any finite flat QMV or $\sqrt{'}$ QMV algebra.

2 Quasi-MV algebras as labelled MV algebras

In Ledda et al. (2006) it is shown that every QMV algebra can be embedded into the direct product of an MV algebra and a flat QMV algebra. Although this result provides some useful information about the relation between QMV algebras in general and MV algebras, its main drawback seems to be the fact that the given embedding is not necessarily an isomorphism—to be sure, it is never such except in trivial cases. In this section, we try to amend this defect by considering two different representations of QMV algebras as *labelled* MV algebras. In both cases the representation is *exact*, meaning that the embedding function is actually an isomorphism.

2.1 Slice algebras

We start by introducing a construction which extracts, out of any MV algebra, a structure in the similarity type of MV algebras.

Definition 1 Let $\mathbf{A} = \left\langle A, \oplus^{\mathbf{A}}, '^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \right\rangle$ be an MV algebra. Let $\mathbf{X} = \{\langle X_a, j_a, f_a \rangle \mid a \in A\}$ be an indexed family s.t., for every $a \in A$, X_a is a nonempty set, $j_a \in X_a$, and $f_a : X_a \to X_{a'}$ is such that $f_{a'} \circ f_a = id_{X_a}$ and $j_{a'} = f_a(j_a)$. The slice algebra over $\mathbf{A}$ with labels in $\mathbf{X}$ is the algebra

$$S(\mathbf{A}) = \left\langle S(A), \oplus^{S(\mathbf{A})}, '^{S(\mathbf{A})}, 0^{S(\mathbf{A})}, 1^{S(\mathbf{A})} \right\rangle$$

where:

- $S(A) = \{\langle x, a \rangle \mid x \in X_a, a \in A\}$ (*this is disjoint union*);
- $\langle x, a \rangle \oplus^{S(\mathbf{A})} \langle y, b \rangle = \left\langle j_{a \oplus^{\mathbf{A}} b}, a \oplus^{\mathbf{A}} b \right\rangle$;
- $\langle x, a \rangle'^{S(\mathbf{A})} = \left\langle f_a(x), a'^{\mathbf{A}} \right\rangle$;
- $0^{S(\mathbf{A})} = \left\langle j_0, 0^{\mathbf{A}} \right\rangle$;
- $1^{S(\mathbf{A})} = \left\langle j_1, 1^{\mathbf{A}} \right\rangle$.

Example 2 Consider the slice algebra over the standard MV algebra $\mathbf{MV}_{[0,1]}$, with labels in $\mathbf{X} = \left\{\left\langle [0,1]_a, \frac{1}{2}_a, f_a \right\rangle \mid a \in [0,1]\right\}$, where $f_a(b) = 1 - b$. This is a quasi-MV algebra, isomorphic to the standard quasi-MV algebra $\mathbf{S}$ via the mapping $\varphi(\langle a, b \rangle) = \langle b, a \rangle$.

The previous example can be generalised. Indeed—recalling from Definition 1 in Ledda et al. (2006) that the equations A1 to A7 therein axiomatise the variety $\mathbb{QMV}$-it is readily seen that:

Lemma 3 *Every slice algebra $S(\mathbf{A})$ over an MV algebra $\mathbf{A}$ is a quasi-MV algebra.*

Proof A1, A3, A4, A6 hold because of properties of MV algebras. To validate A2, just use the stipulation that $f_{a'} \circ f_a(x) = x$ for every $x \in X_a$ and for every $a \in A$. Likewise, A5 and A7 follow from the fact that $j_{a'} = f_a(j_a)$ for every $a \in A$. □

Remark that, in every slice algebra $S(\mathbf{A})$, $\langle x, a \rangle \leq^{S(\mathbf{A})} \langle y, b \rangle$ iff $a \leq^{\mathbf{A}} b$. We now prove the converse to Lemma 3.

Lemma 4 *Every quasi-MV algebra $\mathbf{A}$ is isomorphic to a slice algebra over $\mathbf{R}_A$, the subalgebra of regular elements of $\mathbf{A}$.*

Proof Consider the slice algebra $S(\mathbf{R}_A)$ with labels in

$$\mathbf{X} = \{\langle \{x \in A \mid x \oplus 0 = a\}, a, f_a \rangle \mid a \in \mathcal{R}(\mathbf{A})\},$$

where $f_a(x) = x'$. Let $h : A \to S(\mathcal{R}(\mathbf{A}))$ be defined, for $b \in A$, by

$$h(b) = \langle b, b \oplus 0 \rangle.$$

An easy check will confirm that h is an isomorphism between $\mathbf{A}$ and $S(\mathbf{R}_A)$. □

2.2 Numbered MV algebras

Another way to tag elements of MV algebras in order to characterise any quasi-MV algebra up to isomorphism proceeds by recourse to cardinal numbers. After recalling that, in any MV algebra, inverse admits at most one fixpoint, we are ready to introduce the notions of *labelling function* and *size function* over an MV algebra.

Definition 5 Let $\mathbf{A}$ be an MV algebra. A *labelling function* on $\mathbf{A}$ is a function l such that:

- if $a \in A$ and $a \neq a'$, then $l(a)$ is a cardinal number and $l(a) = l(a')$;
- if $a \in A$ and $a = a'$, then $l(a)$ is an ordered pair of cardinal numbers.

Given a labelling function l, a *size function* on $\mathbf{A}$ is a function sz_l from A to the class of cardinal numbers such that (the symbol $+$ denotes the ordinal sum):

$$sz_l(a) = \begin{cases} \rho, & \text{if } l(a) = \rho, \\ \eta + 2\lambda, & \text{if } l(a) = \langle \eta, \lambda \rangle. \end{cases}$$

Definition 6 Let $\mathbf{A}$ be an MV algebra. Fix a labelling function l and the corresponding size function sz_l on $\mathbf{A}$. Moreover, let $*$ be an object which belongs neither to A nor to the class of ordinal numbers. A *numbered MV algebra* over **A** is an algebra of the form

$$N(\mathbf{A}) = \left\langle N(A), \oplus^{N(\mathbf{A})}, '^{N(\mathbf{A})}, 0^{N(\mathbf{A})}, 1^{N(\mathbf{A})} \right\rangle$$

where:

- $N(A) = \{\langle x, a\rangle \mid x \in sz_l(a) \text{ or } x = *, \text{ and } a \in A\}$
- $\langle x, a\rangle \oplus^{N(\mathbf{A})} \langle y, b\rangle = \left\langle *, a \oplus^{\mathbf{A}} b\right\rangle$,
- if $l(a) = \rho$, then $\langle x, a\rangle'^{N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}}\right\rangle$,
- if $l(a) = \langle \eta, \lambda\rangle$ then $\langle *, a\rangle'^{N(\mathbf{A})} = \left\langle *, a'^{\mathbf{A}}\right\rangle$,
- if $l(a) = \langle \eta, \lambda\rangle$ and $x < \eta$, then $\langle x, a\rangle'^{N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}}\right\rangle$,
- if $l(a) = \langle \eta, \lambda\rangle$ and $\eta \leq x < \eta + \lambda$ then $\langle x, a\rangle'^{N(\mathbf{A})} = \left\langle x + \lambda, a'^{\mathbf{A}}\right\rangle$,
- if $l(a) = \langle \eta, \lambda\rangle$ and $\eta \leq x < \eta+\lambda$ then $\langle x + \lambda, a\rangle'^{N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}}\right\rangle$.

Again, a numbered MV algebra turns out to be a QMV algebra.

Lemma 7 *Any numbered MV algebra is a quasi-MV algebra.*

Proof We confine ourselves to show that inverse is involutive. In fact, if $l(a) = \langle \eta, \lambda\rangle$ and $x < \eta$, then $\langle x, a\rangle'^{N(\mathbf{A})\prime N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}}\right\rangle'^{N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}\prime\mathbf{A}}\right\rangle = \langle x, a\rangle$, whereas if $l(a) = \langle \eta, \lambda\rangle$ and $\eta \leq x < \eta+\lambda$, then $\left\langle x, a\right\rangle'^{N(\mathbf{A})\prime N(\mathbf{A})} = \left\langle x+\lambda, a'^{\mathbf{A}}\right\rangle'^{N(\mathbf{A})} = \left\langle x, a'^{\mathbf{A}\prime\mathbf{A}}\right\rangle = \left\langle x, a\right\rangle$. The remainder of the proof is left to the reader. □

The converse requires more work. We prove that:

Theorem 8 *Every quasi-MV algebra* **A** *is isomorphic to a numbered MV algebra over* $\mathbf{R_A}$.

Proof Let **A** be a quasi-MV algebra. If $a \in \mathcal{R}(\mathbf{A})$, we define:

$$cl_1(a) = \left\{x \in A \mid a \neq x \text{ and } a/\chi = x/\chi \text{ and } x = x'\right\};$$

$$cl_2(a) = \left\{x \in A \mid a \neq x \text{ and } a/\chi = x/\chi \text{ and } x \neq x'\right\};$$

$$l(a) = \begin{cases} card\,(cl_2(a))\,, \text{ if } a \neq a', \\ \langle card(cl_1(a)), card(cl_2(a))\rangle\,, \text{ if } a = a'. \end{cases}$$

Consider the numbered MV algebra $N(\mathbf{R_A})$ associated with the labelling function l and with the corresponding size function sz_l. For $a \in A$, fix a bijection κ between $(a \oplus 0)/\chi - \{a \oplus 0\}$ and $sz_l(a \oplus 0)$, such that $\kappa(b) < \kappa(c)$ whenever $b = b'$ and $c \neq c'$. We define φ as follows:

$$\varphi(a) = \begin{cases} *, \text{ if } a \in \mathcal{R}(\mathbf{A}), \\ \kappa(a), \text{ otherwise.} \end{cases}$$

Now, let $f : A \rightarrow N(\mathcal{R}(\mathbf{A}))$ be given by:

$$f(a) = \langle \varphi(a), a \oplus 0\rangle\,.$$

The function f is clearly well defined and onto. It is also one–one; in fact, suppose $f(a) = f(b)$. If $a, b \in \mathcal{R}(\mathbf{A})$, then $f(a) = \langle *, a\rangle = \langle *, b\rangle = f(b)$ implies $a = b$. If $a, b \notin \mathcal{R}(\mathbf{A})$, $f(a) = \langle \kappa(a), a \oplus 0\rangle = \langle \kappa(b), b \oplus 0\rangle = f(b)$ means that a and b sit in the same cloud and have the same label, whence they are one and the same element. By definition of $\varphi(a)$ and by Definition 6, the remaining cases cannot arise. We now show that the basic operations are preserved by f.

$$\begin{aligned} f(a) \oplus^{N(\mathbf{R_A})} f(b) &= \left\langle \varphi(a), a \oplus^{\mathbf{A}} 0\right\rangle \oplus^{N(\mathbf{R_A})} \left\langle \varphi(b), b \oplus^{\mathbf{A}} 0\right\rangle \\ &= \left\langle *, a \oplus^{\mathbf{A}} b \oplus^{\mathbf{A}} 0\right\rangle \\ &= f\left(a \oplus^{\mathbf{A}} b\right). \end{aligned}$$

If $a \in \mathcal{R}(\mathbf{A})$, then:

$$\begin{aligned} f(a)'^{N(\mathbf{R_A})} &= \left\langle *, a \oplus^{\mathbf{A}} 0\right\rangle'^{N(\mathbf{R_A})} \\ &= \left\langle *, \left(a \oplus^{\mathbf{A}} 0\right)'^{\mathbf{A}}\right\rangle \\ &= \left\langle *, a'^{\mathbf{A}} \oplus^{\mathbf{A}} 0\right\rangle \\ &= f(a'^{\mathbf{A}}). \end{aligned}$$

If $a \notin \mathcal{R}(\mathbf{A})$ and $a = a'$, then:

$$\begin{aligned} f(a)'^{N(\mathbf{R_A})} &= \left\langle \kappa(a), a \oplus^{\mathbf{A}} 0\right\rangle'^{N(\mathbf{R_A})} \\ &= \left\langle \kappa(a), \left(a \oplus^{\mathbf{A}} 0\right)'^{\mathbf{A}}\right\rangle \\ &= \left\langle \kappa(a), a'^{\mathbf{A}} \oplus^{\mathbf{A}} 0\right\rangle \\ &= f(a'^{\mathbf{A}}). \end{aligned}$$

If $a \notin \mathcal{R}(\mathbf{A})$ and $a \neq a'$, then:

$$\begin{aligned} f(a)'^{N(\mathbf{R_A})} &= \left\langle \kappa(a), a \oplus^{\mathbf{A}} 0\right\rangle'^{N(\mathbf{R_A})} \\ &= \left\langle \kappa(a) + \lambda, \left(a \oplus^{\mathbf{A}} 0\right)'^{\mathbf{A}}\right\rangle \\ &= \left\langle \kappa(a) + \lambda, a'^{\mathbf{A}} \oplus^{\mathbf{A}} 0\right\rangle \\ &= f(a'^{\mathbf{A}}). \end{aligned}$$

□

Example 9 The diamond (Example 3 in Ledda et al. 2006) can be represented as a numbered MV algebra as follows: $0 = \langle *, 0\rangle\,, a = \langle 1, b\rangle\,, b = \langle *, b\rangle\,, 1 = \langle *, 1\rangle$.

In our opinion, approaching quasi-MV algebras via numbered MV algebras is especially interesting since two quasi-MV algebras are isomorphic if, and only if, their associated MV algebras are isomorphic and their labels coincide.

3 Varieties of quasi-MV algebras

The structure of the lattice of subvarieties of $\mathbb{MV}$ is well-known, thanks to a comprehensive study due to Komori (1981), as well as to Di Nola and Lettieri (1999); see also Cignoli et al. (1999). The aim of this section is to investigate as accurately as possible the structure of the lattice $\mathcal{L}^V(\mathbb{QMV})$ of subvarieties of $\mathbb{QMV}$, against the background of Komori's classification of MV algebraic varieties.

For a start, the structure of the "flat" side of $\mathcal{L}^V$ is easily described. In fact:

Lemma 10 *There are just two nontrivial varieties of flat QMV algebras*:

- $\mathbb{FQMV} = \mathbf{V}(\mathbf{F}_{02})$;
- $\mathbf{V}(\mathbf{F}_{10})$*, which is axiomatised relative to* $\mathbb{FQMV}$ *by the equation* $x \approx x'$.

Proof It is easily seen that $\mathbf{F}_{02}$ and $\mathbf{F}_{10}$ are subdirectly irreducible ($\mathbf{F}_{10}$, for that matter, is simple). Moreover, any nontrivial flat QMV algebra contains either $\mathbf{F}_{02}$ or $\mathbf{F}_{10}$ as a subalgebra; thus, for every class $\mathbb{K}$ of flat algebras (containing at least one nontrivial algebra), either $\mathbf{F}_{02} \in \mathbf{S}(\mathbb{K})$ or $\mathbf{F}_{10} \in \mathbf{S}(\mathbb{K})$, whence either $\mathbf{V}(\mathbf{F}_{02}) \subseteq \mathbf{V}(\mathbb{K})$ or $\mathbf{V}(\mathbf{F}_{10}) \subseteq \mathbf{V}(\mathbb{K})$. By adapting the argument of Theorem 60 in Ledda et al. (2006), one sees that $\mathbf{F}_{02}$ generates $\mathbb{FQMV}$ as a variety. It follows that $\mathbf{V}(\mathbf{F}_{10})$ is the unique nontrivial proper subvariety of $\mathbb{FQMV}$, and an argument similar to Theorem 60 in Ledda et al. (2006) shows that it is axiomatised relative to $\mathbb{FQMV}$ by the equation $x \approx x'$. □

Splitting pairs are a useful tool for the structural description of lattices, in particular lattices of subvarieties of a variety (cp. e.g. Kowalski and Ono 2000). We recall that an ordered pair $\langle a, b\rangle$ of elements of a lattice $\mathbf{L}$ is said to *split* $\mathbf{L}$ iff $a \nleq^{\mathbf{L}} b$ and, for any c in L, either $a \leq^{\mathbf{L}} c$ or $c \leq^{\mathbf{L}} b$. In other words, b is the largest element of the lattice not above a. The next lemma identifies two splitting pairs in $\mathcal{L}^V(\mathbb{QMV})$; hereafter, $\mathbb{BA}$ denotes the variety of Boolean algebras.

Lemma 11 *The pairs* $\langle \mathbf{V}(\mathbf{F}_{10}), \mathbb{MV}\rangle$ *and* $\langle \mathbb{BA}, \mathbb{FQMV}\rangle$ *split the lattice* $\mathcal{L}^V(\mathbb{QMV})$.

Proof Let $\mathbb{V} \subsetneq \mathbb{MV}$. Then there exists $\mathbf{A} \in \mathbb{V}$ s.t. $A - \mathcal{R}(\mathbf{A}) \neq \emptyset$. Let θ be the congruence on $\mathbf{A}$ whose sole blocks are $\mathcal{R}(\mathbf{A})$ and $A - \mathcal{R}(\mathbf{A})$. Then $\theta \neq \omega$; therefore, $\mathbf{A}/\theta = \mathbf{F}_{10}$, i.e. $\mathbf{F}_{10} \in \mathbf{H}(\mathbb{V}) = \mathbb{V}$, whereby $\mathbf{V}(\mathbf{F}_{10}) \subseteq \mathbb{V}$. Now, let $\mathbb{V} \subsetneq \mathbb{FQMV}$. Then there exists $\mathbf{A} \in \mathbb{V}$ with at least two clouds. Hence $\mathbf{B}_2$, the two-element Boolean algebra, is a subalgebra of $\mathbf{A}/\chi$, i.e. $\mathbf{B}_2 \in \mathbf{SH}(\mathbb{V}) = \mathbb{V}$, whereby $\mathbf{V}(\mathbf{B}_2) = \mathbb{BA} \subseteq \mathbb{V}$. □

Corollary 12 $\mathbf{V}(\mathbf{F}_{10})$ *and* $\mathbb{BA}$ *are the only atoms of* $\mathcal{L}^V(\mathbb{QMV})$.

Proof Remark that $\mathbf{F}_{10}$ and $\mathbf{B}_2$ are strictly simple QMV algebras (indeed, the only strictly simple QMV algebras). By an argument similar to Theorem 2.1 in Galatos (2005), this implies that they are atoms of $\mathcal{L}^V(\mathbb{QMV})$. That there can be no further atoms is a consequence of Lemma 11. □

Corollary 13 *If* $\mathbb{V}$ *is a variety of quasi-MV algebras, the following are equivalent*:

- $\mathbb{V} \subseteq \mathbb{MV}$;
- $\mathbb{V}$ *is subtractive w.r.t.* 0.

Proof If $\mathbb{V} \subseteq \mathbb{MV}$, then $\mathbb{V}$ is subtractive w.r.t. 0. Conversely, let $\mathbb{V} \subsetneq \mathbb{MV}$. To show that it is not subtractive w.r.t. 0, it is sufficient to prove that $\mathbb{V}$ is not 0-permutable. Anyway, consider $\mathbf{F}_{20} \in \mathbf{V}(\mathbf{F}_{10}) \subseteq \mathbb{V}$, and call $0, a, b$ its elements. The congruences θ, with blocks $\{a\}$ and $\{0, b\}$, and φ, with blocks $\{0\}$ and $\{a, b\}$, are such that $\langle a, 0\rangle \in \varphi \circ \theta$ because $\langle a, b\rangle \in \varphi$ and $\langle b, 0\rangle \in \theta$, but it cannot be the case that $\langle a, 0\rangle \in \theta \circ \varphi$, for $a/\theta \cap 0/\varphi = \emptyset$. □

Subtractivity is by no means the only important algebraic property that is peculiar to MV algebraic subvarieties, in the context of $\mathcal{L}^V(\mathbb{QMV})$. Other examples are provided by properties concerning congruence lattices. It turns out, in fact, that the class of congruence lattices of members of any "pure" subvariety of QMV algebras validates just the equations satisfied by *all* lattices, and nothing else. More precisely:

Theorem 14 *If* $\mathbb{V}$ *is a variety of quasi-MV algebras, the following are equivalent*:

- $\mathbb{V} \subseteq \mathbb{MV}$;
- *There is a nontrivial lattice equation* ϵ *which is satisfied in* $\{\mathcal{C}(\mathbf{A}) : \mathbf{A} \in \mathbb{V}\}$.

Proof If $\mathbb{V} \subseteq \mathbb{MV}$, then $\{\mathcal{C}(\mathbf{A}) : \mathbf{A} \in \mathbb{V}\}$ satisfies e.g. distribution, which is a nontrivial lattice equation.

For the converse, consider any $\mathbf{B} \in \mathbf{V}(\mathbf{F}_{10})$. Any equivalence relation on $\mathbf{B}$ must necessarily respect both truncated sum and inverse; therefore $\mathcal{C}(\mathbf{B}) = \mathrm{Eqv}(\mathbf{B})$. Hence, $\{\mathcal{C}(\mathbf{B}) : \mathbf{B} \in \mathbf{V}(\mathbf{F}_{10})\}$ contains *only* lattices of equivalence relations; but it also contains, up to isomorphism, *all* lattices of equivalence relations, because for any cardinal κ we can construct an algebra $\mathbf{B} \in \mathbf{V}(\mathbf{F}_{10})$ containing just κ many fixpoints for the inverse.

Now, let $\mathbb{V} \subsetneq \mathbb{MV}$. By Lemma 11 $\mathbf{V}(\mathbf{F}_{10}) \subseteq \mathbb{V}$. If ϵ is a nontrivial lattice equation, it has a counterexample in some lattice, and by Whitman's theorem (Theorem 4.62 in McKenzie et al. 1987) it has a counterexample in some lattice of equivalence relations, hence in $\{\mathcal{C}(\mathbf{B}) : \mathbf{B} \in \mathbf{V}(\mathbf{F}_{10})\}$ and therefore in $\{\mathcal{C}(\mathbf{A}) : \mathbf{A} \in \mathbb{V}\}$. □

Remark that we actually proved something stronger than we claimed in the statement of the previous theorem: ϵ needs not actually be an equation, but can be any universal formula—for all universal formulas carry over to subalgebras. More than that, ϵ could be a property which is not even expressible in the language of lattices but which, within the framework of congruence lattices, is known to imply some given nontrivial universal formula—e.g. congruence 3-permutability, which is known to imply congruence modularity. Therefore, we are permitted to strengthen Theorem 14 as follows:

Theorem 15 *If $\mathbb{V}$ is a variety of quasi-MV algebras, the following are equivalent*:

- $\mathbb{V} \subseteq \mathbb{MV}$;
- *There exists some property P which is satisfied in $\{\mathcal{C}(\mathbf{A}) : \mathbf{A} \in \mathbb{V}\}$ and implies some nontrivial universal formula in the language of lattices.*

Corollary 16 *No "genuine" variety of QMV algebras is either congruence distributive, or congruence modular, or congruence n-permutable for any n* (cp. Lipparini et al. 1995), *or e-regular for any constant e*.

Theorem 55 in Ledda et al. (2006) implies that the varietal join $\mathbb{MV} \vee \mathbb{FQMV}$ in $\mathcal{L}^V(\mathbb{QMV})$ is just $\mathbb{QMV}$. What about the binary joins $\mathbb{V} \vee \mathbb{FQMV}$, where $\mathbb{V}$ is a proper subvariety of $\mathbb{MV}$? With the next theorem, we will provide a general answer. Recall the following definition from Chajda (1995):

Definition 17 An equation $t \approx s$ of type τ is called *normal* whenever neither t nor s is a variable or else t and s are the same variable.

For our purposes, we need to adapt the previous definition as follows:

Definition 18 An equation $t \approx s$ of type $\langle 2, 1, 0, 0\rangle$ is called $\oplus-$*normal* whenever each one of t, s either contains no variables or contains at least an occurrence of $\oplus$.

Remark that an equation $t \approx s$ of type $\langle 2, 1, 0, 0\rangle$ is $\oplus-$normal just in case, given any QMV algebra $\mathbf{A}$ and any $\overrightarrow{a}, \overrightarrow{b} \in A$, $t^{\mathbf{A}}(\overrightarrow{a})$ and $s^{\mathbf{A}}(\overrightarrow{b})$ are regular elements of $\mathbf{A}$.

By Komori's classification of MV varieties (see Cignoli et al. 1999, Chap. 8), every proper nontrivial subvariety of $\mathbb{MV}$ is generated by the union of two families $\{\mathbf{Ł}_i\}_{i \in I}$, $\{\mathbf{K}_j\}_{j \in J}$, where the former set contains finite Łukasiewicz chains and each $\mathbf{K}_j$ has the form $\Gamma(\Lambda(\mathbf{Z}), \langle j-1, 0\rangle)$. Furthermore, each such variety is axiomatised relative to $\mathbb{MV}$ by a set of $\oplus-$ normal equations. We have the following result:

Theorem 19 *Let $\mathbb{V}$ be a proper nontrivial subvariety of MV algebras whose equational basis relative to $\mathbb{MV}$ is $\mathcal{E}$. The following subvarieties of $\mathbb{QMV}$ are mutually coincident*:

- $\mathbf{V}(\{\mathbf{A} : \mathbf{A}/\chi \in \mathbb{V}_{SI}\})$;
- $\mathbb{V} \vee \mathbb{FQMV}$;
- $\mathrm{mod}(\mathcal{E})$;
- $\mathbf{V}(\{\mathbf{A} \times \mathbf{F}_{02} : \mathbf{A} \in \mathbb{V}_{SI}\})$.

Proof We first show that $\mathbf{V}(\{\mathbf{A} : \mathbf{A}/\chi \in \mathbb{V}_{SI}\}) \subseteq \mathbb{V} \vee \mathbb{FQMV}$. If $\mathbf{A}$ is such that $\mathbf{A}/\chi \in \mathbb{V}_{SI}$, Theorem 55 in Ledda et al. (2006) implies that $\mathbf{A} \in \mathbf{SP}(\mathbb{V}_{SI} \cup \mathbb{FQMV})$, whence

$$\begin{aligned}\mathbf{V}(\{\mathbf{A} : \mathbf{A}/\chi \in \mathbb{V}_{SI}\}) \subseteq\ & \mathbf{HSPSP}(\mathbb{V}_{SI} \cup \mathbb{FQMV}) \\ =\ & \mathbf{HSP}(\mathbf{V}(\mathbb{V}_{SI}) \cup \mathbb{FQMV}) \\ =\ & \mathbf{HSP}(\mathbb{V} \cup \mathbb{FQMV}) \\ =\ & \mathbb{V} \vee \mathbb{FQMV}.\end{aligned}$$

To see that $\mathbb{V} \vee \mathbb{FQMV} \subseteq \mathrm{mod}(\mathcal{E})$, it is sufficient to remark that $\mathbb{V}$ satisfies $\mathcal{E}$ by assumption, while flat QMV algebras satisfy any $\oplus-$normal equation.

We prove that $\mathrm{mod}(\mathcal{E}) \subseteq \mathbf{V}(\{\mathbf{A} : \mathbf{A}/\chi \in \mathbb{V}_{SI}\})$. Let $\mathbf{A}$ be a subdirectly irreducible QMV algebra which satisfies $\mathcal{E}$, i.e. $\mathbf{A} \in (\mathrm{mod}(\mathcal{E}))_{SI}$. Then $\mathbf{A}/\chi \in (\mathrm{mod}(\mathcal{E} \cup \{x \approx x \oplus 0\}))_{SI}$, that is to say $\mathbf{A}/\chi \in \mathbb{V}_{SI}$.

To show that $\mathbb{V} \vee \mathbb{FQMV} \subseteq \mathbf{V}(\{\mathbf{A} \times \mathbf{F}_{02} : \mathbf{A} \in \mathbb{V}_{SI}\})$, we establish the contrapositive. Suppose the equation $t \approx s$ fails in $\mathbb{V} \vee \mathbb{FQMV}$. Then, either it fails in some subdirectly irreducible member of $\mathbb{V}$, whence it fails in $\{\mathbf{A} \times \mathbf{F}_{02} : \mathbf{A} \in \mathbb{V}_{SI}\}$, or it fails in $\mathbb{FQMV}$, whence it fails in $\mathbf{F}_{02}$ and thus in $\{\mathbf{A} \times \mathbf{F}_{02} : \mathbf{A} \in \mathbb{V}_{SI}\}$.

Finally, $\mathbf{V}(\{\mathbf{A} \times \mathbf{F}_{02} : \mathbf{A} \in \mathbb{V}_{SI}\}) \subseteq \mathbf{V}(\{\mathbf{A} : \mathbf{A}/\chi \in \mathbb{V}_{SI}\})$ follows from the fact that $\mathbf{A} \times \mathbf{F}_{02}/\chi$ is isomorphic to $\mathbf{A}$ and thus, if the latter belongs to $\mathbb{V}_{SI}$, so does the former. □

Some special instances of the previous theorem are worth emphasising. Recall the next definition from Ledda et al. (2006).

Definition 20 A QMV algebra $\mathbf{A}$ is called *irreducible* iff $\mathbf{A}/\chi = \mathbf{B}_2$. We denote by $\mathbb{I}$ the class of irreducible QMV algebras.

The class $\mathbb{I}$ is a universal class, axiomatized by the first order formulas

$$0 \neq 1$$
$$\forall x (x \oplus 0 \approx 0 \vee x \oplus 0 \approx 1)$$

but it is not a variety or even a quasivariety, since it is closed neither w.r.t. products ($\mathbf{B}_2$ is irreducible, while $\mathbf{B}_2 \times \mathbf{B}_2$ has four clouds) nor w.r.t. quotients (if $\mathbf{A}$ is any irreducible QMV algebra, $\mathbf{A}/\tau$ is flat). Upon observing that $\mathbf{B}_2$ is the sole subdirectly irreducible Boolean algebra and that $x \Cap x' \approx 0$ is a $\oplus-$normal equation constituting an equational basis for $\mathbb{BA}$ relative to $\mathbb{MV}$, we can exploit Theorem 19 to describe the variety generated by irreducible QMV algebras:

Corollary 21 *The following subvarieties of $\mathbb{QMV}$ are mutually coincident*:

- $\mathbf{V}(\mathbb{I})$;
- $\mathbb{BA} \vee \mathbb{FQMV}$;
- $\mathrm{mod}(x \Cap x' \approx 0)$;
- $\mathbf{V}(\mathbf{B}_2 \times \mathbf{F}_{02})$.

The last corollary admits a generalisation which does not entirely follow from Theorem 19. We define:

Definition 22 A QMV algebra $\mathbf{A}$ is called *n-irreducible* iff $\mathbf{A}/\chi = \mathbf{Ł}_n$. We denote by $\mathbb{I}^n$ the class of n-irreducible QMV algebras.

Obviously, irreducible QMV algebras are the same as 2-irreducible QMV algebras.

Theorem 23 *For $n > 2$, the following subvarieties of $\mathbb{QMV}$ are mutually coincident*:

- $\mathbf{V}\left(\left\{\mathbb{I}^j : j \leq n\right\}\right)$;
- $\mathbb{MV}_n \vee \mathbb{FQMV}$:
- $\mathrm{mod}((n)x \approx (n-1)x)$;
- $\mathbf{V}\left(\left\{\mathbf{Ł}_j \times \mathbf{F}_{02} : j \leq n\right\}\right)$.

Proof The proof is almost exactly as in Theorem 19. To see that $\mathrm{mod}((n)x \approx (n-1)x) \subseteq \mathbf{V}(\mathbb{I}^n)$, just remark that if $\mathbf{A}$ is subdirectly irreducible then it is totally preordered, by the subdirect representation theorem for QMV algebras; so, if it satisfies $(n)x \approx (n-1)x$, then its subalgebra of regular elements is in $\left\{\mathbf{Ł}_j : j \leq n\right\}$ and thus $\mathbf{A}$ is in $\left\{\mathbb{I}^j : j \leq n\right\}$. □

We close this section with a further observation concerning binary joins between subvarieties of $\mathbb{MV}$ and subvarieties of $\mathbf{V}(\mathbf{F}_{10})$. For this purpose, we need the following

Definition 24 Let t be a term of type $\langle 2, 1, 0, 0\rangle$. The term $\varphi(t)$ is so inductively defined:

- if t is a variable or a constant, $\varphi(t) = t \oplus 0$;
- if $t = s'$, then $\varphi(t) = \varphi(s)'$;
- if $t = s_1 \oplus s_2$, then $\varphi(t) = \varphi(s_1) \oplus \varphi(s_2)$.

Lemma 25 *Let t be a term of type $\langle 2, 1, 0, 0\rangle$ which contains at least an occurrence of $\oplus$. Then any subvariety $\mathbb{V}$ of $\mathbb{QMV}$ satisfies the equation $t \approx \varphi(t)$.*

Proof Induction on the complexity of t. Since t contains at least an occurrence of $\oplus$, it cannot be an atomic term; its minimum possible complexity is therefore represented by the case $t = s_1 \oplus s_2$, where each s_i is either a variable or a constant—and our claim trivially follows. Now, let our claim hold whenever the complexity of a term is less than n, and let t have complexity n. If $t = s'$, then for any $\overrightarrow{a} \in \mathbf{A} \in \mathbb{V}$ we have that $\varphi(t)^{\mathbf{A}}(\overrightarrow{a}) = \varphi(s')^{\mathbf{A}}(\overrightarrow{a}) = \left(\varphi(s)^{\mathbf{A}}(\overrightarrow{a})\right)' = s^{\mathbf{A}}(\overrightarrow{a})' = t^{\mathbf{A}}(\overrightarrow{a})$. If $t = s_1 \oplus s_2$, then for any $\overrightarrow{a} \in \mathbf{A} \in \mathbb{V}$ we have that $\varphi(t)^{\mathbf{A}}(\overrightarrow{a}) = \varphi(s_1 \oplus s_2)^{\mathbf{A}}(\overrightarrow{a}) = \varphi(s_1)^{\mathbf{A}}(\overrightarrow{a}) \oplus \varphi(s_2)^{\mathbf{A}}(\overrightarrow{a}) = s_1^{\mathbf{A}}(\overrightarrow{a}) \oplus s_2^{\mathbf{A}}(\overrightarrow{a}) = t^{\mathbf{A}}(\overrightarrow{a})$. □

We are now in a position to show, for an $\oplus$-normal equation $t \approx s$ which holds in a given variety of MV algebras, that it continues to hold if the latter is joined with $\mathbb{FQMV}$.

Lemma 26 *Let $t \approx s$ be an $\oplus$-normal equation of type $\langle 2, 1, 0, 0\rangle$ satisfied by $\mathbb{V} \subseteq \mathbb{MV}$. Then $\mathbb{V} \vee \mathbb{FQMV}$ satisfies $t \approx s$ as well.*

Proof Obviously $\mathbb{V}$ satisfies $\varphi(t) \approx \varphi(s)$. Now, let $\mathbf{A} \in \mathbb{V}_{SI}$ and let $\overrightarrow{\langle a, b\rangle} \in A \times F_{02}$; then $\varphi(t)^{\mathbf{A} \times \mathbf{F}_{02}}(\overrightarrow{\langle a, b\rangle}) = \left\langle c, 0^{\mathbf{F}_{02}}\right\rangle$ for some c in A, and similarly for interpretations of $\varphi(s)$. Thus $\mathbf{A} \times \mathbf{F}_{02}$ satisfies $\varphi(t) \approx \varphi(s)$ and so, by Theorem 19, $\mathbb{V} \vee \mathbb{FQMV}$ satisfies $\varphi(t) \approx \varphi(s)$. If both t and s contain $\oplus$, we have our conclusion applying Lemma 25. If either t or s is a constant, the conclusion follows by recalling that $0 \oplus 0 = 0$ and $1 \oplus 0 = 1$ in any QMV algebra. □

Theorem 27 *For $\mathbb{V} \subseteq \mathbb{MV}$ ($\mathbb{V}$ nontrivial), $\mathbb{V} \vee \mathbf{V}(\mathbf{F}_{10}) = \mathbb{V} \vee \mathbb{FQMV}$.*

Proof For the nontrivial direction, let t, s be terms of type $\langle 2, 1, 0, 0\rangle$ s.t. $\mathbb{V} \vee \mathbf{V}(\mathbf{F}_{10})$ satisfies $t \approx s$, whence in particular $\mathbb{V}$ satisfies $t \approx s$. If such an equation is $\oplus$-normal, then $\mathbb{V} \vee \mathbb{FQMV}$ satisfies it by Lemma 26. Otherwise, remark that it cannot be the case that t is a variable followed by an odd number of occurrences of $'$, while at the same time s is the same variable followed by an even number of occurrences of $'$, for such an equation renders trivial any subvariety of $\mathbb{MV}$ which satisfies it, against our hypothesis. Since equations of the remaining forms which hold in $\mathbb{V} \vee \mathbf{V}(\mathbf{F}_{10})$ must also hold in $\mathbb{V} \vee \mathbb{FQMV}$, we have our conclusion. □

4 Ideals of quasi-MV algebras

In Ledda et al. (2006) we remarked that $\mathbb{QMV}$ fails to be an ideal determined variety; we also noticed that the class $\mathcal{I}(\mathbf{A})$ of ideals (defined in analogy with MV algebras) of a given QMV algebra $\mathbf{A}$ cannot, as a rule, be bijectively mapped onto the class of its congruences. The purpose of the present short section is showing that, although congruences in a QMV algebra are generally determined neither by such ideals nor by ideals in the sense of Gumm and Ursini (1984), these classes are mutually distinct.

We start by recalling some relevant universal algebraic notions. We assume that the language of the algebras mentioned in what follows contains a constant 0.

Definition 28 Let $\mathbb{K}$ be a class of similar algebras. A $n+m$ -ary term

$p(x_1, \ldots, x_n, y_1, \ldots, y_m)$

in the language of $\mathbb{K}$ is called a $\mathbb{K}$-*ideal term in* $y_1, \ldots, y_m$ iff

$$\mathbb{K} \vDash p(x_1, \ldots, x_n, 0, \ldots, 0) \approx 0.$$

We denote by $IT_{\mathbb{K}}(y_1, \ldots, y_m)$ the set of all $\mathbb{K}$-ideal terms in $y_1, \ldots, y_m$.

Definition 29 Gumm and Ursini (1984) Let $\mathbf{A}$ be a member of $\mathbb{K}$ and let $\emptyset \neq J \subseteq A$. J is called a $\mathbb{K}$-*ideal* of $\mathbf{A}$ iff for any $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in $IT_{\mathbb{K}}(y_1, \ldots, y_m)$, for any $a_1, \ldots, a_n \in A$ and for any $b_1, \ldots, b_m \in J$, we have $p^{\mathbf{A}}(a_1, \ldots, a_n, b_1, \ldots, b_m) \in J$.

For $\mathbf{A}$ a member of $\mathbb{K}$, we denote by $\mathcal{I}_{\mathbb{K}}(\mathbf{A})$ the lattice of all $\mathbb{K}$-ideals of $\mathbf{A}$, as well as, by a notational abuse, its universe. If $H \subseteq A$, moreover, $\langle H]_{\mathbf{A}}$ is the smallest $\mathbb{K}$-ideal of $\mathbf{A}$ containing H; outer brackets are omitted if H is a singleton.

Any 0-coset of some congruence on a given algebra is usually called a *normal set* for that algebra. The set of all normal sets for an algebra $\mathbf{A}$ in $\mathbb{K}$ may or may not coincide with the set of all $\mathbb{K}$-ideals of $\mathbf{A}$. Whenever this happens for all $\mathbf{A} \in \mathbb{K}$, the following definition applies.

Definition 30 Ursini (1994) A class of similar algebras $\mathbb{K}$ is said to have *normal ideals* iff for any $\mathbf{A}$ in $\mathbb{K}$ we have that

$$\mathcal{I}_{\mathbb{K}}(\mathbf{A}) = \{0/\theta : \theta \in \mathcal{C}(\mathbf{A})\}.$$

We now recall the notions of IC-system and of finitely congruential variety (Aglianò and Ursini 1997; cp. also Spinks 2003).

Definition 31 Let $\mathbb{K}$ be a class of similar algebras, and let $P = \{p_i(x, y) : i \in I\}$ be an indexed set of binary terms in the language of $\mathbb{K}$. If $\mathbf{A} \in \mathbb{K}$ and $J \in \mathcal{I}_{\mathbb{K}}(\mathbf{A})$, then we set

$$J^P = \left\{\langle a, b\rangle \in A^2 : p_i^{\mathbf{A}}(a, b) \in J \text{ for every } i \in I\right\}.$$

Definition 32 Let $\mathbb{K}$ be a class of similar algebras. An indexed set $P = \{p_i(x, y) : i \in I\}$ of binary terms in the language of $\mathbb{K}$ is called an *IC-system for* $\mathbb{K}$ iff for all $\mathbf{A}$ in $\mathbb{K}$:

- $(0/\theta)^P \in \mathcal{C}(\mathbf{A})$;
- $0/_{(0/\theta)^P} = 0/\theta$.

Definition 33 A variety $\mathbb{V}$ is called *finitely congruential* iff it has a finite IC system.

Definition 34 Let $\mathbf{A}$ be an algebra and let $J \in \mathcal{I}_{\mathbb{K}}(\mathbf{A})$. We define:

$$\theta^J = \bigvee \{\phi \in \mathcal{C}(\mathbf{A}) : 0/\phi = J\}$$

In the following, we will need the results listed hereafter:

Theorem 35 *For a variety* $\mathbb{V}$:

- *if it is finitely congruential and it has normal ideals then it is subtractive* (Aglianò and Ursini 1997, *Theorem 3.10*);
- *if it has normal ideals and* $\{\mathbf{A} : \mathbf{A} \simeq \mathbf{B}/\theta^{\langle 0]_{\mathbf{B}}}$ *for some* $\mathbf{B}$ *in* $\mathbb{V}\}$ *is a quasivariety, then it is finitely congruential* (Aglianò and Ursini 1997, *Theorem 3.13*; Spinks 2003, *Theorem 1.8.14*).

So much for the necessary algebraic preliminaries. Now we prove a useful property of the congruence χ in QMV algebras.

Lemma 36 *Let* $\mathbf{A}$ *be a QMV algebra. Then*

$$\chi = \bigvee \{\phi \in \mathcal{C}(\mathbf{A}) : 0/\phi = cl(0)\}.$$

Proof Obviously χ is such that $0/\chi = cl(0)$. Thus, what we must show is that if $0/\theta = cl(0)$, then $\theta \subseteq \chi$. Suppose $\langle a, b\rangle \in \theta$ and $0/\theta = cl(0)$. Then, since $b \otimes b' = 0$, $\langle a \otimes b', 0\rangle \in \theta$ and so $a \otimes b' \in cl(0)$. Since $a \otimes b'$ is regular, this means that $a \otimes b' = 0$, i.e. $a \leq b$. Similarly, $b \leq a$, whence $\langle a, b\rangle \in \chi$. □

Finally, here is the main result of this section.

Theorem 37 *Let* $\mathbf{A}$ *be a QMV algebra. Then* $\mathcal{I}(\mathbf{A}) \neq \mathcal{I}_{\mathbb{QMV}}(\mathbf{A})$.

Proof Suppose otherwise, and let $J \in \mathcal{I}(\mathbf{A})$. Then, by Theorem 45.3 in Ledda et al. (2006), $J = g(f(J)) = 0/f(J)$, i.e.

$$\mathcal{I}_{\mathbb{QMV}}(\mathbf{A}) = \mathcal{I}(\mathbf{A}) \subseteq \{0/\theta : \theta \in \mathcal{C}(\mathbf{A})\}.$$

Since $\{0/\theta : \theta \in \mathcal{C}(\mathbf{A})\} \subseteq \mathcal{I}_{\mathbb{QMV}}(\mathbf{A})$ always holds, it follows that $\mathbb{QMV}$ has normal ideals. On the other hand, we also have that

$$\begin{aligned}\theta^{\langle 0]_{\mathbf{A}}} &= \bigvee\{\phi \in \mathcal{C}(\mathbf{A}) : 0/\phi = \langle 0]_{\mathbf{A}}\} && \text{(by Definition 34)}\\ &= \bigvee\{\phi \in \mathcal{C}(\mathbf{A}) : 0/\phi = cl(0)\} && \text{(by the absurdum hyp.)}\\ &= \chi && \text{(by Lemma 36)}\end{aligned}$$

Thus, putting $\mathbb{K} = \{\mathbf{A} : \mathbf{A} \simeq \mathbf{B}/\theta^{\langle 0]_{\mathbf{B}}}$ for some $\mathbf{B}$ in $\mathbb{QMV}\}$, it follows that

$$\mathbb{K} = \{\mathbf{A} : \mathbf{A} \simeq \mathbf{B}/\chi \text{ for some } \mathbf{B} \text{ in } \mathbb{QMV}\} = \mathbb{MV}.$$

So $\mathbb{K}$ would be a variety, whence $\mathbb{QMV}$ would be a finitely congruential variety by Theorem 35 and a subtractive one by the same theorem. This, however, contradicts Corollary 33 in Ledda et al. (2006). □

5 Generators for the quasivarieties $\mathbb{QMV}$ and $\sqrt{'}\,\mathbb{QMV}$

In Paoli et al. (submitted) one finds theorems to the effect that $\mathbb{QMV}$ and $\sqrt{'}\,\mathbb{QMV}$ are generated, as varieties, by their finite members. The same paper raises the open question whether such classes are generated by their finite members also as *quasivarieties* (a property known as *strong finite model property*). In the present subsection we answer this question in the affirmative as regards $\mathbb{QMV}$. We also prove that the quasivariety $\mathbb{C}$ of Cartesian algebras is generated by the standard algebra $\mathbf{S}_r$, while the set $\{\mathbf{S}_r, \mathbf{F}_{124}\}$ is a set of generators for $\sqrt{'}\ \mathbb{QMV}$, again taken as a quasivariety.

As a first step, we establish the strong finite model property for flat QMV algebras.

Lemma 38 *The variety* $\mathbb{FQMV}$ *has the strong finite model property.*

Proof In order to prove that $\mathbb{FQMV}$ is generated as a quasivariety by its finite members, we make use of the Grätzer–Lakser result according to which $\mathbf{Q}(\mathbb{K}) = \mathbf{ISP}(\mathbb{K})$ whenever $\mathbb{K}$ is a finite set of finite algebras (Grätzer and Lakser 1973), and show that $\mathbf{ISP}(\mathbf{F}_{12}) = \mathbb{FQMV}$. More precisely, remark that quasiequations can only contain finitely many variables and thus, if at all, they can be falsified in a finitely generated—hence countable—algebra. Therefore, restriction to countable algebras entails no loss of generality. We prove that every countable flat QMV algebra can be embedded into a suitable direct power of $\mathbf{F}_{12}$.

Let $F_{12} = \{0, a, a', b\}$. If $\mathbf{C}$ is a countable flat QMV algebra, we start by partitioning its universe into two disjoint subsets,

$$C_1 = \{c \in C : c = c'\};$$
$$C_2 = \{c \in C : c \neq c'\}.$$

Next, we order C_1 arbitrarily, and C_2 in such a way that, for every $i \in N$, $c_i' = c_{i+1}$ or $c_i' = c_{i-1}$. For any $0 \neq c_i \in C_1$, let $f(c_i)$ be the sequence

$$\langle \underbrace{0, \ldots, 0}_{i-1 \text{ times}}, b, 0, \ldots \rangle$$

Moreover, for any $c_j \in C_2$, let $f(c_j)$ be either the sequence

$$\langle \underbrace{0, \ldots, 0}_{card(C_1)+j-1 \text{ times}}, a, a', 0, \ldots \rangle$$

or the sequence

$$\langle \underbrace{0, \ldots, 0}_{card(C_1)+j-2 \text{ times}}, a', a, 0, \ldots \rangle$$

according as c_j immediately precedes or immediately follows c_j' in the enumeration of C_2. Finally, let $f(0)$ be the sequence consisting of all 0's. It is easily checked that f embeds $\mathbf{C}$ into the direct power $\mathbf{F}_{12}^{\omega}$. □

A result in Blok and Ferreirim (2000) implies that MV algebras have the strong finite model property. As a consequence, we can use the direct decomposition theorem for $\mathbb{QMV}$ to attain our goal.

Theorem 39 *The variety* $\mathbb{QMV}$ *has the strong finite model property.*

Proof Let $\&_{i \leq n} t_i \approx s_i \Rightarrow t \approx s$ be a quasiequation which fails in $\mathbb{QMV}$. Recalling that quasiequations carry over to subalgebras and products, by the direct decomposition theorem for $\mathbb{QMV}$ there are an MV algebra $\mathbf{M}$ and a flat QMV algebra $\mathbf{F}$ s.t. $\&_{i \leq n} t_i \approx s_i \Rightarrow t \approx s$ fails in $\mathbf{M} \times \mathbf{F}$, hence either in $\mathbf{M}$ or in $\mathbf{F}$. If the former, then our quasiequation fails in a finite member of $\mathbb{MV}$ by the strong finite model property for this variety; if the latter, our result follows from Lemma 38. □

Let us now take into account $\sqrt{'}$ QMV algebras. Here, we confine ourselves to showing that flat algebras have the strong finite model property.

Lemma 40 *The variety* $\mathbb{F}$ *has the strong finite model property.*

Proof In order to prove that $\mathbb{F}$ is generated as a quasivariety by its finite members, we show that $\mathbf{ISP}(\mathbf{F}_{124}) = \mathbb{F}$. As in Lemma 38, restriction to countable algebras entail no loss of generality. We thus prove that every countable flat QMV algebra can be embedded into a suitable direct power of $\mathbf{F}_{124}$.

Let the universe of $\mathbf{F}_{124}$ be

$$\left\{0, b, a, a', c, \sqrt{'}c, c', \sqrt{'}c'\right\}.$$

If $\mathbf{C}$ is a countable flat $\sqrt{'}$QMV algebra, we partition its universe into three disjoint subsets,

$$P_1 = \left\{p \in C : p = \sqrt{'}p\right\};$$
$$P_2 = \left\{p \in C : p \neq \sqrt{'}p \text{ and } p = p'\right\};$$
$$P_3 = \left\{p \in C : p \neq \sqrt{'}p \text{ and } p \neq p'\right\}.$$

Next, we order P_1 arbitrarily, P_2 in such a way that, for every $i \in N$, $\sqrt{'}p_i = p_{i+1}$ or $\sqrt{'}p_i = p_{i-1}$, and P_3 in such a way that the members of each 4-element "windmill" are listed together in a row, next to one another. For any $p_i \in P_1$, let $f(p_i)$ be the sequence

$$\langle \underbrace{0, \ldots, 0}_{i-1 \text{ times}}, b, 0, \ldots \rangle$$

For any $p_j \in P_2$, let $f(p_j)$ be either the sequence

$$\langle \underbrace{0, \ldots, 0}_{card(P_1)+j-1 \text{ times}}, a, \sqrt{'}a, 0, \ldots \rangle$$

or the sequence

$$\left\langle \underbrace{0, \ldots, 0}_{card(P_1)+j-2 \text{ times}}, \sqrt{\prime}a, a, 0, \ldots \right\rangle$$

according as p_j immediately precedes or immediately follows $\sqrt{\prime}p_j$ in the enumeration of P_2. Finally, for any $p_k \in P_3$, let $f(p_k)$ be one of the following sequences:

$$\left\langle \underbrace{0, \ldots, 0}_{card(P_1 \cup P_2)+j-1 \text{ times}}, c, \sqrt{\prime}c, c', \sqrt{\prime}c', 0, \ldots \right\rangle$$

$$\left\langle \underbrace{0, \ldots, 0}_{card(P_1 \cup P_2)+j-2 \text{ times}}, \sqrt{\prime}c, c', \sqrt{\prime}c', c, 0, \ldots \right\rangle$$

$$\left\langle \underbrace{0, \ldots, 0}_{card(P_1 \cup P_2)+j-3 \text{ times}}, c', \sqrt{\prime}c', c, \sqrt{\prime}c, 0, \ldots \right\rangle$$

$$\left\langle \underbrace{0, \ldots, 0}_{card(P_1 \cup P_2)+j-4 \text{ times}}, \sqrt{\prime}c', c, \sqrt{\prime}c, c', 0, \ldots \right\rangle$$

according as p_k sits in first, second, third or fourth position within its own "windmill". It is easily checked that f embeds **C** into the direct power $\mathbf{F}_{124}^{\omega}$. □

Lewin et al. (2004) showed that much of the structure theory for MV algebras, including an appropriate version of Chang's completeness theorem, carries over to MV algebras with an element $k = k'$ which realises a constant in the signature. Throughout this section, therefore, the term "MV algebras" will refer to such expanded structures.

Definition 41 Let $\mathbb{K}$ be a class of MV algebras. By $\mathcal{P}(\mathbb{K})$ we mean the class of all pair algebras over algebras in $\mathbb{K}$.

Since MV algebras and $\sqrt{\prime}$ QMV algebras have different signatures, we will henceforth append subscripts to class operators. The absence of subscripts will indicate that the operator at issue acts upon algebras of the same signature as MV algebras, while a $\sqrt{\prime}$ as a subscript will mean that the operator affects algebras in the language of $\sqrt{\prime}$ QMV algebras.

Lemma 42 *Let* $\mathbb{K}$ *be a class of MV algebras. Then*: (i) $\mathcal{P}\mathbf{S}(\mathbb{K}) \subseteq \mathbf{S}_{\sqrt{\prime}}\mathcal{P}(\mathbb{K})$*;* (ii) $\mathcal{P}\mathbf{P}(\mathbb{K}) \subseteq \mathbf{P}_{\sqrt{\prime}}\mathcal{P}(\mathbb{K})$*;* (iii) $\mathcal{P}\mathbf{P}_U(\mathbb{K}) \subseteq \mathbf{P}_{U\sqrt{\prime}}\mathcal{P}(\mathbb{K})$.

Proof (i) has been proved in Paoli et al. (submitted), Lemma 48. (ii) We must prove that, if $\mathbf{A}_i \in \mathbb{K}$ for each $i \in I$, then $\mathcal{P}(\Pi_{i\in I}\mathbf{A}_i)$ is isomorphic to $\Pi_{i\in I}\mathcal{P}(\mathbf{A}_i)$. So, let $\varphi : \mathcal{P}(\Pi_{i\in I}\mathbf{A}_i) \to \Pi_{i\in I}\mathcal{P}(\mathbf{A}_i)$ be given by

$$\varphi\left(\left\langle\left\langle \ldots, a_i^1, \ldots\right\rangle, \left\langle \ldots, a_i^2, \ldots\right\rangle\right\rangle\right) = \left\langle \ldots, \left\langle a_i^1, a_i^2\right\rangle, \ldots\right\rangle.$$

This function is clearly onto. It is one–one, as

$$\left\langle \ldots, \left\langle a_i^1, a_i^2\right\rangle, \ldots\right\rangle = \left\langle \ldots, \left\langle b_i^1, b_i^2\right\rangle, \ldots\right\rangle$$

implies that, for each i, $a_i^1 = b_i^1$ and $a_i^2 = b_i^2$, whence $\langle\langle \ldots, a_i^1, \ldots\rangle, \langle \ldots, a_i^2, \ldots\rangle\rangle = \langle\langle \ldots, b_i^1, \ldots\rangle, \langle \ldots, b_i^2, \ldots\rangle\rangle$. The constants are obviously preserved; thus, what remains to be shown is preservation of both $\oplus$ and $\sqrt{\prime}$. As regards $\oplus$,

$$\begin{aligned}
&\varphi\left(\left\langle\left\langle .., a_i^1, ..\right\rangle, \left\langle .., a_i^2, ..\right\rangle\right\rangle \oplus \left\langle\left\langle .., b_i^1, ..\right\rangle, \left\langle .., b_i^2, ..\right\rangle\right\rangle\right)\\
&= \varphi\left(\left\langle\left\langle .., a_i^1 \oplus b_i^1, ..\right\rangle, \langle .., k, ..\rangle\right\rangle\right)\\
&= \left\langle .., \left\langle a_i^1 \oplus b_i^1, k\right\rangle, ..\right\rangle\\
&= \left\langle .., \left\langle a_i^1, a_i^2\right\rangle, ..\right\rangle \oplus \left\langle .., \left\langle b_i^1, b_i^2\right\rangle, ..\right\rangle\\
&= \varphi\left(\left\langle\left\langle .., a_i^1, ..\right\rangle, \left\langle .., a_i^2, ..\right\rangle\right\rangle\right)\\
&\oplus \varphi\left(\left\langle\left\langle .., b_i^1, ..\right\rangle, \left\langle .., b_i^2, ..\right\rangle\right\rangle\right)
\end{aligned}$$

Finally, as regards $\sqrt{\prime}$,

$$\begin{aligned}
\varphi\left(\sqrt{\prime}\left(\left\langle\left\langle .., a_i^1, ..\right\rangle, \left\langle .., a_i^2, ..\right\rangle\right\rangle\right)\right) &= \varphi\left(\left\langle\left\langle .., a_i^2, ..\right\rangle, \left\langle .., a_i^{1\prime}, ..\right\rangle\right\rangle\right)\\
&= \left\langle .., \left\langle a_i^2, a_i^{1\prime}\right\rangle, ..\right\rangle\\
&= \sqrt{\prime}\left\langle .., \left\langle a_i^1, a_i^2\right\rangle, ..\right\rangle\\
&= \sqrt{\prime}\varphi\left(\left\langle\left\langle .., a_i^1, ..\right\rangle, \left\langle .., a_i^2, ..\right\rangle\right\rangle\right).
\end{aligned}$$

(iii) It suffices to prove that, if $\mathbf{A}_m \in \mathbb{K}$ for each $m \in M$ (for M an ultrafilter of a given set of indices I), then $\mathcal{P}(\Pi_{i\in M}\mathbf{A}_m)$ is isomorphic to $\Pi_{i\in M}\mathcal{P}(\mathbf{A}_m)$. An argument similar to the above one shows that this is indeed the case. □

Lemma 43 *The standard algebra* $\mathbf{S}_r$ *generates* $\mathbb{C}$ *as a quasivariety.*

Proof By Di Nola's representation theorem for MV algebras, every MV algebra **A** belongs to $\mathbf{ISPP}_U\left(\mathbf{MV}_{[0,1]}\right)$. Remark that such a result holds whether the algebra is required to contain a fixpoint k (which realises a constant included in the signature) or not, so from now on we will assume that it does. It follows that, for every MV algebra **A**, $\mathcal{P}(\mathbf{A}) \in \mathcal{P}\mathbf{ISPP}_U\left(\mathbf{MV}_{[0,1]}\right)$ and, by Lemma 42—omitting subscripts —$\mathcal{P}(\mathbf{A}) \in \mathbf{ISPP}_U\mathcal{P}\left(\mathbf{MV}_{[0,1]}\right) = \mathbf{ISPP}_U(\mathbf{S}_r)$. On the other hand, every cartesian $\sqrt{\prime}$ QMV algebra is embeddable into a pair algebra over some MV algebra, whence every cartesian $\sqrt{\prime}$ QMV algebra belongs to $\mathbf{ISPP}_U(\mathbf{S}_r)$. □

Theorem 44 *The pair* $\{\mathbf{S}_r, \mathbf{F}_{124}\}$ *is a set of generators for the quasivariety* $\sqrt{\prime}\mathbb{QMV}$.

Proof By Lemmas 40 and 43, $\mathbb{C} \cup \mathbb{F} \subseteq \mathbf{ISPP}_U(\mathbf{S}_r, \mathbf{F}_{124})$. Since $\sqrt{'}\mathbb{QMV} \subseteq \mathbf{SP}(\mathbb{C} \cup \mathbb{F})$, we have that

$$\begin{aligned}\sqrt{'}\mathbb{QMV} \subseteq \mathbf{SP}(\mathbb{C} \cup \mathbb{F}) &\subseteq \mathbf{SPISPP}_U(\mathbf{S}_r, \mathbf{F}_{124}) \\ &= \mathbf{ISPSPP}_U(\mathbf{S}_r, \mathbf{F}_{124}) \\ &\subseteq \mathbf{ISSPPP}_U(\mathbf{S}_r, \mathbf{F}_{124}) \\ &\subseteq \mathbf{ISPP}_U(\mathbf{S}_r, \mathbf{F}_{124})\end{aligned}$$

□

6 Amalgamation property

According to the website *Mathematical Structures* (http://math.chapman.edu/cgi-bin/structures.pl?HomePage), "An amalgam is a tuple $(\mathbf{A}, f, \mathbf{B}, g, \mathbf{C})$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are structures of the same signature, and $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ are embeddings (injective morphisms). A class $\mathbb{K}$ of structures is said to have the *amalgamation property* if for every amalgam with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{K}$ and $A \neq \emptyset$ there exists a structure $\mathbf{D} \in \mathbb{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ f$." A couple of decades ago, Mundici (1987) proved that MV algebras have the amalgamation property Mundici (1987). It is all too natural, therefore, to inquire whether QMV algebras have the property too. The present section is devoted to showing that they do; we also prove that both cartesian and flat $\sqrt{'}$ QMV algebras amalgamate.

6.1 Amalgamation in quasi-MV algebras

The amalgamation property for QMV algebras is a useful application of Theorem 8, since it relies in an essential way on the representation of QMV algebras as numbered MV algebras.

Theorem 45 $\mathbb{QMV}$ *has the amalgamation property.*

Proof Let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{QMV}$ where $h_i : \mathbf{A} \to \mathbf{B}_i$, for $i \in \{1, 2\}$, is an embedding. To show that $\mathbb{QMV}$ has the amalgamation property it is enough to construct a quasi-MV algebra $\mathbf{D}$ and to provide embeddings $g_i : \mathbf{B}_i \to \mathbf{D}$, for $i \in \{1, 2\}$, such that for each $a \in A$, $g_1(h_1(a)) = g_2(h_2(a))$.

Now, the restrictions $h_i \lceil \mathcal{R}(\mathbf{A})$ are embeddings of $\mathbf{R_A}$ into $\mathbf{R}_{\mathbf{B}_i}$, for $a = a \oplus^{\mathbf{A}} 0$ implies $h_i(a) = h_i(a \oplus^{\mathbf{A}} 0) = h_i(a) \oplus^{\mathbf{B}_i} 0$. Therefore, the quintuple

$$\left\langle \mathbf{R_A}, h_1 \lceil \mathcal{R}(\mathbf{A}), \mathbf{R}_{\mathbf{B}_1} h_2 \lceil \mathcal{R}(\mathbf{A}), \mathbf{R}_{\mathbf{B}_2} \right\rangle$$

is an amalgam and, since $\mathbb{MV}$ has the amalgamation property, there exist an MV algebra $\mathbf{C}$ and embeddings $w_i : \mathbf{R}_{\mathbf{B}_i} \to \mathbf{C}$ s.t., for any $a \in \mathcal{R}(\mathbf{A})$, $w_1(h_1(a)) = w_2(h_2(a))$. We are left with the task of extending this property to any (not necessarily regular) $a \in A$. In virtue of Theorem 8, we can, up to isomorphism, uniquely associate to $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ their corresponding numbered MV algebras $N(\mathbf{A})$ **,** $N(\mathbf{B}_1)$, $N(\mathbf{B}_2)$, with respective labelling functions $l_{\mathbf{A}}, l_{\mathbf{B}_i}$, and use them in establishing the amalgamation property for $\mathbb{QMV}$. Let ρ_i be an abbreviation for $l_{\mathbf{B}_i}(w_i^{-1}(a))$. Construct the numbered MV algebra $N(\mathbf{C}) = \left\langle N(C), \oplus^{N(\mathbf{C})}, {}'^{N(\mathbf{C})}, 0^{N(\mathbf{C})}, 1^{N(\mathbf{C})} \right\rangle$ where the labelling function $l_{\mathbf{C}}$ is such that, for any $a \in C$,

$$l_{\mathbf{C}}(a) = \begin{cases} \max(\rho_1, \rho_2), \text{ if } \rho_1, \rho_2 \text{ are cardinals;} \\ \langle \max(\pi_1(\rho_1), \pi_1(\rho_2)), \max(\pi_2(\rho_1), \pi_2(\rho_2)) \rangle \\ \quad , \text{ if } \rho_1, \rho_2 \text{ are ordered pairs of cardinals.} \end{cases}$$

Notice that crossover cases cannot arise: being embeddings, the w_i's map fixpoints to fixpoints and conversely. Now, let $g_i : N(\mathbf{B}_i) \to N(\mathbf{C})$, for $i \in \{1, 2\}$, be given, for any $\langle x_i, b\rangle \in N(B_i)$, by $g_i(\langle x, b\rangle) = \langle x, w_i(b)\rangle$. These functions are well-defined and injective. In fact, suppose $g_i(\langle x, a\rangle) = g_i(\langle y, b\rangle)$; this implies $\langle x, w_i(a)\rangle = \langle y, w_i(b)\rangle$, whence $x = y$ and $a = b$, since w_i is, by assumption, an embedding. Moreover g_i preserves the basic operations. We confine ourselves to the case of truncated sum (a routinary check takes care of inverse).

$$\begin{aligned} g_i\left(\langle x, a\rangle \oplus^{N(\mathbf{B}_i)} \langle y, b\rangle\right) &= g_i\left(\left\langle *, a \oplus^{\mathbf{B}_i} b\right\rangle\right) \\ &= \left\langle *, w_i\left(a \oplus^{\mathbf{B}_i} b\right)\right\rangle \\ &= \left\langle *, w_i(a) \oplus^{\mathbf{C}} w_i(b)\right\rangle \\ &= g_i(\langle x, a\rangle) \oplus^{N(\mathbf{C})} g_i(\langle y, b\rangle) \end{aligned}$$

Finally, it has to be proved that, for $\langle x, a\rangle \in N(A)$, $g_1(h_1\langle x, a\rangle) = g_2(h_2\langle x, a\rangle)$. More precisely, if f is the isomorphism of Theorem 8, we have that:

$$\begin{aligned} g_1(f(h_1(a))) &= g_1(\langle x, h_1(a) \oplus 0\rangle) \\ &= g_1(\langle x, h_1(a \oplus 0)\rangle) \\ &= \langle x, w_1(h_1(a \oplus 0))\rangle \\ &= \langle x, w_2(h_2(a \oplus 0))\rangle \\ &= g_2(\langle x, h_2(a \oplus 0)\rangle) \\ &= g_2(\langle x, h_2(a) \oplus 0\rangle) \\ &= g_2(f(h_2(a))). \end{aligned}$$

□

6.2 Amalgamation in $\sqrt{'}$ quasi-MV algebras

As regards flat $\sqrt{'}$ QMV algebras, we prove a stronger result. Due to the behaviour of truncated sum in $\mathbb{F}$, flat algebras amalgamate if and only if their monounary reducts do. We start with a definition.

Definition 46 We call a monounary algebra $\mathbf{A} = \left\langle A, *^{\mathbf{A}} \right\rangle$ *cyclic* if and only if, for any $x \in A$, there exists a natural number n such that $*^{\mathbf{A}}_{n\text{-times}} x = x$.

Clearly, the monounary reducts of flat $\sqrt{'}$ QMV algebras are cyclic (their cycles being of length at most 4). We have that:

Theorem 47 *Every cyclic monounary algebra has the amalgamation property.*

Proof Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be cyclic monounary algebras and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be embeddings. Note, first of all, that $f(\mathbf{A})$ is a subalgebra of $\mathbf{B}$ isomorphic to $g(\mathbf{A})$ because of the cyclicity of $\mathbf{A}$, and the fact that f and g are embeddings. We now need a monounary algebra $\mathbf{D}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ s.t. $f' \circ f(A) = g' \circ g(A)$. Take $D = (B - f(A)) \cup C$, with

$$*^{\mathbf{D}}(a) = \begin{cases} *^{\mathbf{B}}(a) \text{ if } a \in B, \\ *^{\mathbf{C}}(a) \text{ otherwise.} \end{cases}$$

and define

$$f'(a) = \begin{cases} f(a) \ \text{ if } a \in f(A) \\ a \text{ otherwise} \end{cases}$$

and, analogously,

$$g'(a) = \begin{cases} g(a) \ \text{ if } a \in g(A) \\ a \text{ otherwise.} \end{cases}$$

It is quite immediate that f', g' are embeddings and $f' \circ f(\mathbf{A}) = g' \circ g(\mathbf{A})$. □

Corollary 48 $\mathbb{F}$ *has the amalgamation property.*

We now turn to cartesian algebras and show the same result.

Theorem 49 *The quasivariety* $\mathbb{C}$ *has the amalgamation property.*

Proof Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be cartesian $\sqrt{'}$ QMV-algebras. Let $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ be embeddings. As in Theorem 45, the quintuple $\langle \mathbf{R_A}, h_1 \lceil \mathcal{R}(\mathbf{A}), \mathbf{R_{B_1}} h_2 \lceil \mathcal{R}(\mathbf{A}), \mathbf{R_{B_2}} \rangle$ is an amalgam. So, let $\mathbf{D}$ be the MV-amalgam of $\mathbf{R_B}$, $\mathbf{R_C}$, and let $f'_{\mathcal{R}} : \mathbf{R_B} \to \mathbf{D}$, $g'_{\mathcal{R}} : \mathbf{R_C} \to \mathbf{D}$ be appropriate embeddings. Clearly $f'_{\mathcal{R}} \circ f(\mathbf{R_A}) = g'_{\mathcal{R}} \circ g(\mathbf{R_A})$; nonetheless, remark that although such mappings have to preserve the MV operations, they need not preserve $\sqrt{'}$. Now, let $\pi_{\mathbf{A}} : \mathbf{A} \to \mathcal{P}(\mathbf{R_A})$ be such that, for $a \in A$,

$$\pi_{\mathbf{A}}(a) = \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle,$$

and consider the mappings $p : \mathcal{P}(\mathbf{R_A}) \to \mathcal{P}(\mathbf{R_B})$ and $q : \mathcal{P}(\mathbf{R_A}) \to \mathcal{P}(\mathbf{R_C})$, defined as follows:

$$p(\langle a, b \rangle) = \langle f(a), f(b) \rangle;$$
$$q(\langle a, b \rangle) = \langle g(a), g(b) \rangle;.$$

p and q are seen to be embeddings. Let us now construct the pair algebra $\mathcal{P}(\mathbf{D})$ over $\mathbf{D}$, and take $f' : \mathcal{P}(\mathbf{R_B}) \to \mathcal{P}(\mathbf{D})$ and $g' : \mathcal{P}(\mathbf{R_C}) \to \mathcal{P}(\mathbf{D})$ be such that, for $b \in B$ and $c \in C$,

$$f'(\langle b_1, b_2 \rangle) = \left\langle f'_{\mathcal{R}}(b_1), f'_{\mathcal{R}}(b_2) \right\rangle;$$
$$g'(\langle c_1, c_2 \rangle) = \left\langle g'_{\mathcal{R}}(c_1), g'_{\mathcal{R}}(c_2) \right\rangle.$$

We check that $f' \circ p \circ \pi_{\mathbf{A}} = g' \circ q \circ \pi_{\mathbf{A}}$. In fact,

$$\begin{aligned} & f' \circ p \left(\left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle \right) \\ & = f' \left\langle f(a \oplus 0), f\left(\sqrt{'}a \oplus 0\right) \right\rangle \\ & = \left\langle f'_{\mathcal{R}}(f(a \oplus 0)), f'_{\mathcal{R}}\left(f\left(\sqrt{'}a \oplus 0\right)\right) \right\rangle \\ & = \left\langle g'_{\mathcal{R}}(g(a \oplus 0)), g'_{\mathcal{R}}\left(g\left(\sqrt{'}a \oplus 0\right)\right) \right\rangle \\ & = g' \circ q \left(\left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle \right). \end{aligned}$$

It is immediate to see that $\oplus$ is preserved by $f' \circ p$ and $g' \circ q$ alike. Let us take care of $\sqrt{'}$.

$$\begin{aligned} & f' \circ p \left(\left\langle \sqrt{'}a \oplus 0, \sqrt{'}\sqrt{'}a \oplus 0 \right\rangle \right) \\ & = f' \circ p \left(\left\langle \sqrt{'}a \oplus 0, (a \oplus 0)' \right\rangle \right) \\ & = f' \left\langle f\left(\sqrt{'}a \oplus 0\right), f\left((a \oplus 0)'\right) \right\rangle \\ & = \left\langle f'_{\mathcal{R}}\left(f_{\mathcal{R}}\left(\sqrt{'}a \oplus 0\right)\right), f'_{\mathcal{R}}(f_{\mathcal{R}}((a \oplus 0))' \right\rangle \\ & = \sqrt{'}\left(f' \circ p \left(\left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle \right)\right). \end{aligned}$$

□

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